

WEAK MULTIPLIER BIMONOIDS

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ABSTRACT. Based on the novel notion of ‘weakly counital fusion morphism’, regular weak multiplier bimonoids in braided monoidal categories are introduced. They generalize weak multiplier bialgebras over fields [4] and multiplier bimonoids in braided monoidal categories [5]. Under some assumptions the so-called base object of a regular weak multiplier bimonoid is shown to carry a coseparable comonoid structure; hence to possess a monoidal category of bicomodules. In this case, appropriately defined modules over a regular weak multiplier bimonoid are proven to constitute a monoidal category with a strict monoidal forgetful type functor to the category of bicomodules over the base object.

Braided monoidal categories considered include various categories of modules or graded modules, the category of complete bornological spaces, and the category of complex Hilbert spaces and continuous linear transformations.

1. INTRODUCTION

Hopf algebras can be used to describe symmetries in various situations. Classically, they are vector spaces equipped with the additional structures of a compatible algebra and a coalgebra, and they have categories of representations (modules or comodules) with a monoidal structure, strictly preserved by the forgetful functor to vector spaces. There are Tannaka-style results which allow a Hopf algebra to be reconstructed from its monoidal category of representations together with the forgetful functor.

Various applications lead one to consider more general monoidal categories, which nonetheless share many features with those of the previous paragraph. One can then ask whether they might be the categories of representations of some object more general than a Hopf algebra in the classical sense. It turns out that this is often the case. This is perhaps reminiscent of non-commutative geometry, where non-commutative algebras can sometimes be seen as algebras of functions on some (hypothetical) “non-commutative spaces”. Or for another example, the category of sheaves on a topological space is a topos; but any (Grothendieck) topos can be seen as a generalized space (or as the category of sheaves on a generalized space).

Returning to Hopf algebras, one direction of generalization is where the representations involve an underlying object more general, or simply different, than a vector space. Examples might include modules over commutative rings, graded vector spaces, Hilbert spaces, or bornological vector spaces [15, 18]. A unified treatment of all these situations is possible using the notion of Hopf monoid in a braided monoidal category.

There is another direction of generalization, in which the base objects remain vector spaces, but one generalizes to structures which are not Hopf algebras. For example, functions on a finite group, with values in a field, constitute a Hopf algebra. But if the group is no longer finite, then the functions of finite support form neither an algebra (there is no unit for the pointwise multiplication) nor a coalgebra (the comultiplication which is dual to

the group multiplication does not land in the tensor square of the vector space of finitely supported functions). To axiomatize this situation, the notion of *multiplier Hopf algebra* over a field was proposed by Van Daele in [22].

Yet another generalization of Hopf algebras involves aspects of both of these types of generalization, and proved important in the study of fusion categories [14]. The basic examples of fusion categories are categories of representations of (semi-simple) Hopf algebras. In a fusion category, every irreducible object has an associated ‘dimension’, and in the Hopf algebra case this dimension is always an integer, but in general this need not be the case. It turns out that these non-integral cases can be seen as categories of representations of *weak Hopf algebras* [9, 19]. A weak Hopf algebra is both an algebra and a coalgebra, but the compatibility conditions between these structures are weaker than in an ordinary Hopf algebra, reflecting the fact that the forgetful functor from the category of representations to vector spaces is no longer strict monoidal. But there is a strict monoidal forgetful functor from the category of representations of the weak Hopf algebra to the category of bimodules over a certain separable Frobenius algebra, determined by the weak Hopf algebra and called the *base object*. Note, however, that the monoidal category of bimodules over the base object is not braided, so that this does not reduce to the earlier generalization of Hopf monoids in a braided monoidal category.

A common generalization of multiplier Hopf algebras and weak Hopf algebras was proposed by Van Daele and Wang in [24, 25] under the name *weak multiplier Hopf algebra*.

The axioms of a Hopf algebra, and all of the generalizations listed above, include the existence of a so-called antipode. Omitting this requirement one obtains the more general notion of bialgebra and its various generalizations. The category of representations of a bialgebra is still monoidal, with a strict monoidal forgetful functor to the base category; what is lost in the absence of an antipode is the ability to lift the closed structure of the base monoidal category to the category of representations.

Weak multiplier bialgebras and *multiplier bialgebras* over vector spaces were defined and analyzed in [4]. Their representations are certain non-degenerate modules, and once again the category of representations has a monoidal structure, not preserved by the forgetful functor to vector spaces. But as in the case of weak Hopf algebras, there is still a strict monoidal forgetful functor to an intermediate monoidal category. This time, rather than the monoidal category of bimodules over a separable Frobenius algebra, it is the monoidal category of bicomodules over a coseparable coalgebra constructed from the weak multiplier bialgebra, once again called the base object.

We have recently begun the large program of studying all of these generalizations of Hopf algebras, along with their categories of representations, not just over vector spaces but in more general braided monoidal categories. In [5] we defined multiplier bimonoids in any braided monoidal category. Under further assumptions, we constructed a monoidal category of representations of a multiplier bimonoid. We further developed the theory of multiplier bimonoids in two subsequent papers [6, 7]. Then in [8] we turned to multiplier Hopf monoids; that is, multiplier bimonoids with a suitable antipode map. The present work can be seen as the next step of this program where we generalize the regular weak multiplier bialgebras of [4] to more general braided monoidal categories.

The outline of the paper is as follows. In Section 2 we define the central notion of the paper, that of *regular weak multiplier bimonoid*; this in turn depends on the notion of *weakly counital fusion morphism*, also defined in Section 2. We also discuss various duality principles for these structures, arising from symmetry properties of the axioms. In Section 3 we

study the base objects, and under appropriate assumptions we show that they admit coseparable comonoid structure. While in the category \mathbf{Vect} of vector spaces this was done [4] under the assumption that the comultiplication is full, here we use a different assumption, which follows from fullness of the comultiplication in the case of \mathbf{Vect} . In Section 4 we define and study the category of modules over a regular weak multiplier bimonoid. Under favorable conditions we prove that it is monoidal, via a monoidal structure lifted from the category of bicomodules over the base object. Again, in contrast to [4], this is done assuming not fullness but a substitute which follows from it in the category of vector spaces. We do not address here the analogous question about comodules over a regular weak multiplier bimonoid. We also do not discuss the notion of antipode on a weak multiplier bimonoid (and its bearing on the structure of the category of modules); that is, we do not study weak multiplier Hopf monoids. Section 6 is devoted to the study of the particular case when the braided monoidal base category is also *closed*, as is the case in most of our examples of interest, but not in the case of Hilbert spaces. We discuss consequences of closedness on the assumptions and constructions of the previous sections.

Acknowledgement. We thank Ralf Meyer and Christian Voigt for highly enlightening discussions about bornological vector spaces. We gratefully acknowledge the financial support of the Hungarian Scientific Research Fund OTKA (grant K108384), of ‘Ministerio de Economía y Competitividad’ and ‘Fondo Europeo de Desarrollo Regional FEDER’ (grant MTM2013-41992-P), as well as the Australian Research Council Discovery Grant (DP130101969) and an ARC Future Fellowship (FT110100385). GB expresses thanks for the kind invitations and the warm hospitality that she experienced visiting the University of Granada in Nov 2014, Feb 2015 and Oct 2015. SL is grateful for the warm hospitality of his hosts during visits to the Wigner Research Centre in Sept-Oct 2014 and Aug-Sept 2015.

2. THE AXIOMS

The subject of this section is the introduction of the central notion of the paper: *regular weak multiplier bimonoid*.

Throughout, we work in a braided monoidal category \mathcal{C} . We do not assume that its monoidal structure is strict but — relying on coherence — we omit explicit mention of the associativity and unit isomorphisms. The composite of morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ will be denoted by $g.f: A \rightarrow C$. Any identity morphism will be denoted by 1 . The monoidal product will be denoted by juxtaposition, the monoidal unit by I , and the braiding by c . For n copies of the same object A , we also use the power notation $AA \dots A = A^n$. The same category \mathcal{C} with the reversed monoidal product but the same braiding will be denoted by \mathcal{C}^{rev} . The same monoidal category with the inverse braiding will be denoted by $\overline{\mathcal{C}}$. Performing both these “dualities” (in either order) gives a braided monoidal category $\overline{\mathcal{C}}^{\text{rev}}$.

By a *semigroup* in a braided monoidal category \mathcal{C} we mean a pair (A, m) consisting of an object A of \mathcal{C} and a morphism $m: A^2 \rightarrow A$ — called the *multiplication* — which obeys the associativity condition $m.m1 = m.1m$. The existence of a unit is not required.

Recall from [6] the notion of \mathbb{M} -morphism $X \twoheadrightarrow A$ for a semigroup A and an arbitrary object X . This means a pair $XA \xrightarrow{f_1} A \xleftarrow{f_2} AX$ of morphisms in \mathbb{C} making commutative

$$\begin{array}{ccc} AXA & \xrightarrow{1_{f_1}} & A^2 \\ f_2 1 \downarrow & & \downarrow m \\ A^2 & \xrightarrow{m} & A. \end{array} \quad (2.1)$$

As will be explained in Section 6, \mathbb{M} -morphisms $X \rightarrowtail A$ are related to morphisms from X to the multiplier monoid of A whenever the latter is available.

We say that a morphism $v: ZV \rightarrow W$ in \mathbb{C} is *non-degenerate on the left with respect to some class \mathcal{Y} of objects in \mathbb{C}* if the map

$$C(X, VY) \rightarrow C(ZX, WY), \quad g \mapsto ZX \xrightarrow{1g} ZVY \xrightarrow{v1} WY$$

is injective for any object X , and any object Y in \mathcal{Y} . Symmetrically, v is said to be *non-degenerate on the right with respect to the class \mathcal{Y}* if $v.c$ is non-degenerate on the left with respect to \mathcal{Y} . The multiplication $A^2 \multimap A$ of a semigroup A is termed *non-degenerate with respect to \mathcal{Y}* if it is non-degenerate with respect to \mathcal{Y} both on the left and the right. If a morphism is (left or right) non-degenerate with respect to the one-element class $\{I\}$ then we simply call it *non-degenerate*.

If $XA \dashv_{f_1} A \dashv_{f_2} AX$ is an \mathbb{M} -morphism, and the multiplication of A is non-degenerate with respect to some class \mathcal{V} , then f_1 is non-degenerate on the right with respect to \mathcal{V} if and only if f_2 is non-degenerate on the left with respect to \mathcal{V} . Moreover, the following diagrams commute (see [6]).

$$\begin{array}{ccc} XA^2 & \xrightarrow{1m} & XA \\ f_1 1 \downarrow & & \downarrow f_1 \\ A^2 & \xrightarrow{m} & A \end{array} \quad \begin{array}{ccc} A^2X & \xrightarrow{m1} & AX \\ 1f_2 \downarrow & & \downarrow f_2 \\ A^2 & \xrightarrow{m} & A \end{array} \quad (2.2)$$

Throughout the paper, string diagrams will be used to denote morphisms in braided monoidal categories. In order to give a gentle introduction to their use, in the following definition we use both commutative diagrams and string diagrams in parallel to present the axioms — this is meant to serve as the *Rosetta-stone*.

Definition 2.1. A *weakly counital fusion morphism* in a braided monoidal category \mathcal{C} is an object A equipped with three morphisms $t: A^2 \rightarrow A^2$ (called the *fusion morphism*), $e: A^2 \rightarrow A^2$, and $j: A \rightarrow I$ (called the *counit*). We introduce the morphism

$$m := A^2 \xrightarrow{t} A^2 \xrightarrow{j^1} A \qquad \text{Y-junction} := \text{Y-junction with } t \text{ on the left}$$

and impose the following axioms.

Axiom I. The morphism t obeys the fusion equation:

$$\begin{array}{ccccccc}
 A^3 & \xrightarrow{1t} & A^3 & \xrightarrow{c1} & A^3 & \xrightarrow{1t} & A^3 \xrightarrow{c^{-1}1} A^3 \\
 t1 \downarrow & & & & & & \downarrow t1 \\
 A^3 & \xrightarrow{\hspace{2cm}} & A^3 & & & & \\
 & & 1t & & & &
 \end{array}$$

Axiom II. The morphism e is idempotent:

$$\begin{array}{ccc}
 A^2 & \xrightarrow{e} & A^2 \\
 & \searrow e & \downarrow e \\
 & & A^2
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{Two circles labeled } e \text{ stacked vertically.} \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{One circle labeled } e. \\ \text{Two vertical lines passing through it.} \end{array}
 \end{array}$$

Axiom III. The morphism t is invariant under post-composition by e :

$$\begin{array}{ccc}
 A^2 & \xrightarrow{t} & A^2 \\
 & \searrow t & \downarrow e \\
 & & A^2
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{A circle labeled } t \text{ above a circle labeled } e. \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{A circle labeled } t. \\ \text{Two vertical lines passing through it.} \end{array}
 \end{array}$$

Axiom IV. The following composite morphism is invariant under pre-composition by e :

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{t1} & A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{m1} & A^2 \\
 1e \downarrow & & & & & & & & \parallel \\
 A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{t1} & A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{m1} & A^2
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{A circle labeled } e \text{ above a circle labeled } t. \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{A circle labeled } t. \\ \text{Two vertical lines passing through it.} \end{array}
 \end{array}$$

Axiom V. The following commutativity relation holds between t and e :

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1t} & A^3 \\
 e1 \downarrow & & \downarrow e1 \\
 A^3 & \xrightarrow{1t} & A^3
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{A circle labeled } t \text{ above a circle labeled } e. \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{A circle labeled } e \text{ above a circle labeled } t. \\ \text{Two vertical lines passing through them.} \end{array}
 \end{array}$$

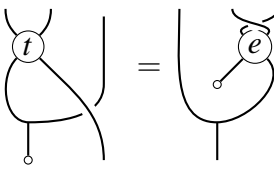
Axiom VI. And also the following commutativity relation holds between t and e :

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{e1} & A^3 & \xrightarrow{1c} & A^3 \\
 t1 \downarrow & & & & & & \downarrow t1 \\
 A^3 & \xrightarrow{1e} & A^3 & & & &
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{A circle labeled } t \text{ above a circle labeled } e. \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{A circle labeled } e \text{ above a circle labeled } t. \\ \text{Two vertical lines passing through them.} \end{array}
 \end{array}$$

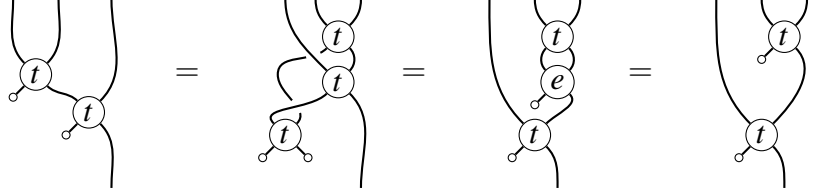
Axiom VII. The following compatibility relation holds between t and e :

$$\begin{array}{ccc}
 A^3 & \xrightarrow{t1} & A^3 & \xrightarrow{1j1} & A^2 \\
 1c \downarrow & & & & \downarrow m \\
 A^3 & & & & \\
 1e \downarrow & & & & \\
 A^3 & \xrightarrow{11j} & A^2 & \xrightarrow{m} & A
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram: } \begin{array}{c} \text{A circle labeled } t \text{ above a circle labeled } e. \\ \text{Two vertical lines passing through them.} \end{array} = \begin{array}{c} \text{A circle labeled } e \text{ above a circle labeled } t. \\ \text{Two vertical lines passing through them.} \end{array}
 \end{array}$$

Axiom VIII. And also the following compatibility relation holds between t and e :

$$\begin{array}{ccc}
A^3 & \xrightarrow{t1} & A^3 & \xrightarrow{1c} & A^3 \\
1c \downarrow & & & & \downarrow m1 \\
A^3 & & & & A^2 \\
1e \downarrow & & & & \downarrow j1 \\
A^3 & \xrightarrow{1j1} & A^2 & \xrightarrow{m} & A
\end{array}$$


It follows by Axioms **I**, **VIII** and **III** that for a weakly counital fusion morphism (A, t, e, j) , $m := j1.t : A^2 \rightarrow A$ is an associative multiplication:



Composing the fusion axiom (Axiom **I**) with $j11$, we obtain the *short fusion equation*:


(2.3)

Some further consequences of the axioms are analyzed in Appendix **A**.

Recall from [5] that a *counital fusion morphism* is a pair consisting of a morphism $t : A^2 \rightarrow A^2$ obeying Axiom **I**, and a morphism $j : A \rightarrow I$ such that $1j.t = 1j$. It is not difficult to see that for any counital fusion morphism (t, j) , there is a weakly counital fusion morphism $(t, 1, j)$. Conversely, if $(t, 1, j)$ is a weakly counital fusion morphism and the induced multiplication $j1.t$ is non-degenerate, then (t, j) is a counital fusion morphism.

It is easy to see that braided strong monoidal functors send weakly counital fusion morphisms to weakly counital fusion morphisms. Since the braiding on \mathcal{C} makes the identity functor into a braided strong monoidal functor $\mathcal{C} \cong \mathcal{C}^{\text{rev}}$, we deduce the following

Lemma 2.2. *For morphisms $t : A^2 \rightarrow A^2$, $e : A^2 \rightarrow A^2$ and $j : A \rightarrow I$ in a braided monoidal category \mathcal{C} , the following assertions are equivalent.*

- (i) *The datum (t, e, j) is a weakly counital fusion morphism in \mathcal{C} .*
- (ii) *The datum $(c^{-1}.t.c, c^{-1}.e.c, j)$ is a weakly counital fusion morphism in \mathcal{C}^{rev} .*

A regular weak multiplier bimonoid in \mathcal{C} will be defined as a quadruple of weakly counital fusion morphisms which obey some compatibility conditions to be discussed next. In developing the theory of regular weak multiplier bimonoids, we shall repeatedly use duality principles in which we move between the braided monoidal category \mathcal{C} , and the related braided monoidal categories \mathcal{C}^{rev} , $\overline{\mathcal{C}}$, and $\overline{\mathcal{C}}^{\text{rev}}$. We shall discuss this more fully below, but for the purpose of the following lemma, we observe that under these duality principles the 4-tuple (t_1, t_2, t_3, t_4) in \mathcal{C} will correspond, respectively, to 4-tuples (t_2, t_1, t_4, t_3) , (t_3, t_4, t_1, t_2) , and (t_4, t_3, t_2, t_1) in the other three categories; furthermore, in the last two cases, the we work with the reverse multiplication $m.c^{-1}$.

Lemma 2.3. *Let A be a semigroup in \mathcal{C} whose multiplication m is non-degenerate with respect to some class containing A . Between some morphisms $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$, consider the following relations.*

$$\begin{array}{c} \text{Diagram (2.4)} \end{array} = \begin{array}{c} \text{Diagram (2.5)} \end{array} \quad (2.4) \quad \begin{array}{c} \text{Diagram (2.5)} \end{array} = \begin{array}{c} \text{Diagram (2.6)} \end{array} \quad (2.5) \quad \begin{array}{c} \text{Diagram (2.6)} \end{array} = \begin{array}{c} \text{Diagram (2.7)} \end{array} \quad (2.6)$$

$$\begin{array}{c} \text{Diagram (2.7)} \end{array} = \begin{array}{c} \text{Diagram (2.8)} \end{array} \quad (2.7) \quad \begin{array}{c} \text{Diagram (2.8)} \end{array} = \begin{array}{c} \text{Diagram (2.9)} \end{array} \quad (2.8) \quad \begin{array}{c} \text{Diagram (2.9)} \end{array} = \begin{array}{c} \text{Diagram (2.10)} \end{array} \quad (2.9)$$

If the identity involving t_i and t_j holds, and the identity involving t_i and t_k holds, for i, j, k different elements of the set $\{1, 2, 3, 4\}$, then also the identity involving t_j and t_k holds.

Proof. We only show three of the implications, all other implications follow symmetrically.

By associativity of the multiplication, (2.4) and (2.5),

$$\begin{array}{c} \text{Diagram (2.7)} \end{array} = \begin{array}{c} \text{Diagram (2.8)} \end{array} = \begin{array}{c} \text{Diagram (2.9)} \end{array} = \begin{array}{c} \text{Diagram (2.10)} \end{array}$$

and now by non-degeneracy of the multiplication this proves (2.7). Similarly, by (2.4), associativity of the multiplication, and (2.6),

$$\begin{array}{c} \text{Diagram (2.7)} \end{array} = \begin{array}{c} \text{Diagram (2.8)} \end{array} = \begin{array}{c} \text{Diagram (2.9)} \end{array} = \begin{array}{c} \text{Diagram (2.10)} \end{array}$$

and now by non-degeneracy of the multiplication this implies (2.8). Finally, by (2.5), (2.6), and associativity of the multiplication,

$$\begin{array}{c} \text{Diagram (2.8)} \end{array} = \begin{array}{c} \text{Diagram (2.9)} \end{array} = \begin{array}{c} \text{Diagram (2.10)} \end{array} = \begin{array}{c} \text{Diagram (2.11)} \end{array}$$

and by non-degeneracy of the multiplication this implies (2.9). \square

Corollary 2.4. *In the setting of Lemma 2.3, the following assertions are equivalent:*

- (i) Conditions (2.4), (2.5) and (2.6) hold;
- (ii) Conditions (2.4), (2.7) and (2.8) hold;
- (iii) Conditions (2.5), (2.7) and (2.9) hold;
- (iv) Conditions (2.6), (2.8) and (2.9) hold.

Corollary 2.5. *If there are morphisms t_1, t_2, t_3, t_4 satisfying the conditions of Corollary 2.4, then any one of them uniquely determines all of the others.*

Lemma 2.6. *Let A be a semigroup in \mathcal{C} whose multiplication is non-degenerate with respect to some class containing A and A^2 . Let $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$ be morphisms satisfying the conditions of Lemma 2.3. The following conditions are equivalent to each other.*

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad (2.10) \qquad \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} \quad (2.11)$$

$$\begin{array}{c} \text{Diagram 5} \end{array} = \begin{array}{c} \text{Diagram 6} \end{array} \quad (2.12) \qquad \begin{array}{c} \text{Diagram 7} \end{array} = \begin{array}{c} \text{Diagram 8} \end{array} \quad (2.13)$$

Proof. Again, we only show that (2.10) implies (2.11), and (2.11) implies (2.13); all other implications follow symmetrically.

By (2.8), (2.10) and (2.8) again,

$$\begin{array}{c} \text{Diagram 9} \end{array} = \begin{array}{c} \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \end{array} = \begin{array}{c} \text{Diagram 12} \end{array}$$

By the non-degeneracy condition on the multiplication this implies (2.11).

Similarly by (2.5), (2.11) and (2.5) again,

$$\begin{array}{c} \text{Diagram 13} \end{array} = \begin{array}{c} \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \end{array} = \begin{array}{c} \text{Diagram 16} \end{array}$$

By the non-degeneracy condition on the multiplication this implies (2.13). \square

Lemma 2.7. *Let A be a semigroup in \mathcal{C} whose multiplication m is non-degenerate with respect to some class containing I and A . Let $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$ be morphisms satisfying the conditions of Lemma 2.3. For any morphism $j: A \rightarrow I$, consider the following conditions.*

$$\begin{array}{c} \text{Diagram 17} \end{array} = \begin{array}{c} \text{Diagram 18} \end{array} \quad (2.14) \qquad \begin{array}{c} \text{Diagram 19} \end{array} = \begin{array}{c} \text{Diagram 20} \end{array} \quad (2.15)$$

$$\begin{array}{c} \text{Diagram 21} \end{array} = \begin{array}{c} \text{Diagram 22} \end{array} \quad (2.16) \qquad \begin{array}{c} \text{Diagram 23} \end{array} = \begin{array}{c} \text{Diagram 24} \end{array} \quad (2.17)$$

Then the following hold.

- (1) Conditions (2.14) and (2.16) are equivalent to each other.
- (2) Conditions (2.15) and (2.17) are equivalent to each other.

Proof. Once again, we only prove that (2.14) implies (2.16); all other implications follow symmetrically.

By (2.5), (2.14), and associativity of the multiplication,

$$\begin{array}{c} \text{Diagram 25} \end{array} = \begin{array}{c} \text{Diagram 26} \end{array} = \begin{array}{c} \text{Diagram 27} \end{array} = \begin{array}{c} \text{Diagram 28} \end{array}$$

By the non-degeneracy condition on the multiplication this implies (2.16). \square

Corollary 2.8. *In the setting of Lemma 2.7, the following assertions are equivalent:*

- (i) Conditions (2.14) and (2.15) hold;
- (ii) Conditions (2.14) and (2.17) hold;
- (iii) Conditions (2.15) and (2.16) hold;
- (iv) Conditions (2.16) and (2.17) hold.

Lemma 2.9. *Let A be a semigroup in \mathcal{C} whose multiplication m is non-degenerate with respect to some class containing I , A and A^2 . Let $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$ be morphisms satisfying the conditions of Lemma 2.3. For a morphism $j: A \rightarrow I$ satisfying the conditions in Lemma 2.7, and morphisms $e_1, e_2: A^2 \rightarrow A^2$ such that*

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}, \quad (2.18)$$

the following assertions are equivalent:

- (i) The morphisms t_1 , j and e_1 constitute a weakly counital fusion morphism in \mathcal{C} ;
- (ii) The morphisms t_2 , j and e_2 constitute a weakly counital fusion morphism in \mathcal{C}^{rev} ;
- (iii) The morphisms t_3 , j and e_2 constitute a weakly counital fusion morphism in $\overline{\mathcal{C}}$;
- (iv) The morphisms t_4 , j and e_1 constitute a weakly counital fusion morphism in $\overline{\mathcal{C}}^{\text{rev}}$.

Proof. We only prove the implications (i) \Rightarrow (ii) and (i) \Rightarrow (iv); the rest of the claims follows symmetrically.

Let us begin with (i) \Rightarrow (iv). The multiplication in part (iv) is the opposite $A^2 \xrightarrow{c^{-1}} A^2 \xrightarrow{m} A$ of the multiplication m in part (i); cf. (2.17). Axiom II in part (iv) has the same form as in part (i). Using (2.6), both sides of Axioms IV, VII, and VIII of part (iv) differ from the respective sides of Axioms IV, VIII, and VII of part (i) by braid isomorphisms. In order to verify the other axioms of part (iv) we have to use the non-degeneracy conditions on the multiplication. Then Axiom I follows by

$$\begin{array}{ccccccc} \text{Diagram 1} & \stackrel{(2.6)}{=} & \text{Diagram 2} & \stackrel{(2.11)}{=} & \text{Diagram 3} & \stackrel{(2.6)}{=} & \text{Diagram 4} \stackrel{A.1}{=} \text{Diagram 5} \stackrel{(2.6)}{=} \\ \text{Diagram 6} & \stackrel{(a)}{=} & \text{Diagram 7} & \stackrel{(2.3)}{=} & \text{Diagram 8} & \stackrel{(2.6)}{=} & \text{Diagram 9} \end{array}$$

where we have written above the equality sign the justification for the given step; the label (a) stands for associativity of the multiplication.

Axiom III follows by

$$\begin{array}{ccccccc} \text{Diagram 1} & \stackrel{A.1}{=} & \text{Diagram 2} & \stackrel{(2.6)}{=} & \text{Diagram 3} & \stackrel{A.1}{=} & \text{Diagram 4} \stackrel{\text{III}}{=} \text{Diagram 5} \stackrel{(2.6)}{=} \text{Diagram 6} \end{array}$$

where the label III refers to the axiom of that number.

Axiom V follows by

and Axiom VI by

The case of assertion (ii) is more involved; non-degeneracy of the multiplication is used in the verification of each axiom. Then Axiom II is immediate by (2.18) and Axiom II of part (i). Axiom I follows by

Axiom III follows by

and Axiom IV by

By (2.18) and (2.4),

Hence Axiom **V** is equivalent to **A.6**; thus it holds true. Similarly, making use of (2.18) and (2.8) Axiom **VI** can be re-written in the equivalent form

which is identity **A.6** for the weakly counital fusion morphism in part (iv); hence it holds true. Axiom **VII** follows by

Finally, Axiom **VIII** follows by

where the unlabelled equality is obtained by applying **A.4** to the weakly counital fusion morphism in part (iv) and using (2.17). \square

Definition 2.10. A *regular weak multiplier bimonoid* in a braided monoidal category \mathcal{C} consists of morphisms $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$, $e_1, e_2: A^2 \rightarrow A^2$ and $j: A \rightarrow I$ satisfying the conditions in Corollary 2.4, Lemma 2.6, Corollary 2.8 and Lemma 2.9 (in particular, non-degeneracy of the multiplication with respect to some class containing I , A and A^2 is required).

The notion of regular weak multiplier bimonoid is invariant under two kinds of symmetry operations:

Corollary 2.11. For morphisms $t_1, t_2, t_3, t_4, e_1, e_2: A^2 \rightarrow A^2$ and $j: A \rightarrow I$ in a braided monoidal category \mathcal{C} , the following assertions are equivalent.

- $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ is a regular weak multiplier bimonoid in \mathcal{C} ;
- $(c^{-1}.t_1.c, c^{-1}.t_2.c, c^{-1}.t_3.c, c^{-1}.t_4.c, c^{-1}.e_1.c, c^{-1}.e_2.c, j)$ is a regular weak multiplier bimonoid in \mathcal{C}^{rev} .

Corollary 2.12. For morphisms $t_1, t_2, t_3, t_4, e_1, e_2: A^2 \rightarrow A^2$ and $j: A \rightarrow I$ in a braided monoidal category \mathcal{C} , the following assertions are equivalent.

- $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ is a regular weak multiplier bimonoid in \mathcal{C} ;
- $(t_2, t_1, t_4, t_3, e_2, e_1, j)$ is a regular weak multiplier bimonoid in \mathcal{C}^{rev} ;
- $(c^{-1}.t_3.c, c^{-1}.t_4.c, c^{-1}.t_1.c, c^{-1}.t_2.c, c^{-1}.e_2.c, c^{-1}.e_1.c, j)$ is a regular weak multiplier bimonoid in $\overline{\mathcal{C}}^{\text{rev}}$;
- $(c^{-1}.t_4.c, c^{-1}.t_3.c, c^{-1}.t_2.c, c^{-1}.t_1.c, c^{-1}.e_1.c, c^{-1}.e_2.c, j)$ is a regular weak multiplier bimonoid in $\overline{\mathcal{C}}$.

We refer to the latter three as the opposite-coopposite, the opposite, and the coopposite of $(t_1, t_2, t_3, t_4, e_1, e_2, j)$.

For a regular weak multiplier bimonoid all identities in Appendix **A** hold, as well as their opposite, coopposite, and opposite-coopposite versions.

Example 2.13. If (t_1, t_2, t_3, t_4, j) is a regular multiplier bimonoid in a braided monoidal category \mathcal{C} in the sense of [5], whose multiplication is non-degenerate with respect to some class containing I , A and A^2 , then with the identity morphism $A^2 \rightarrow A^2$ as e_1 and e_2 it is a regular weak multiplier bimonoid.

Example 2.14. A regular weak multiplier bialgebra over a field, in the sense of [4], is a regular weak multiplier bimonoid in the symmetric monoidal category \mathbf{Vect} of vector spaces, see [1, Theorem 1.2]. However, not every regular weak multiplier bimonoid in \mathbf{Vect} in the sense of Definition 2.10 is a regular weak multiplier bialgebra in the sense of [4, 1], as axiom (vi) in [1, Definition 1.1] is not required in Definition 2.10. (We will consider an appropriate analogue of this ‘missing’ axiom in Section 4.)

3. THE BASE OBJECTS

The key feature of the generalization of bialgebras that are known as *weak bialgebras* is the structure of the categories of modules [10, 2, 20]; this remains true in any braided monoidal category with split idempotents, not just the classical case of vector spaces. As in the case of ordinary bialgebras, these categories of modules are monoidal. However, in contrast to ordinary bialgebras, their monoidal structure is *not* lifted from the base category. The *base object* of a weak bialgebra is both a subalgebra and a quotient coalgebra; we shall usually call it L . These algebra and coalgebra structures obey the compatibility axioms of a *separable Frobenius algebra*. As a consequence, the category of L -bimodules (equivalently, L -bicomodules) is a monoidal category in which the monoidal product is given by splitting a canonical idempotent morphism on the monoidal product of the underlying objects. The category of modules over a weak bialgebra is monoidal via a lifting of this monoidal structure of the category of bi(co)modules over its base object L .

The above properties generalize nicely to *regular weak multiplier bialgebras* over fields [4] with full comultiplication. For such a weak multiplier bialgebra A , the base object L is no longer a subalgebra of A (but it is a non-unital subalgebra of its multiplier algebra). As it has no unit, it can no longer be a separable Frobenius algebra. But it turns out to possess the more general structure of coseparable coalgebra (hence it is a so-called *firm* algebra, see [12]). This structure is enough for the category of L -bicomodules (isomorphically, of firm L -bimodules) to have a monoidal structure where, again, the monoidal product is given by splitting a canonical idempotent morphism. A suitably defined category of A -modules is monoidal via a lifting of this monoidal structure.

The aim of this and of the next sections is to extend the above results to *regular weak multiplier bimonoids* in nice enough braided monoidal categories. We assume that coequalizers exist in our base category \mathcal{C} and that they are preserved by taking the monoidal product with any object; this preservation assumption is automatic if the monoidal category is closed (see Section 6). We will also need the technical assumption that the composite of regular epimorphisms in \mathcal{C} is a regular epimorphism again. We consider regular weak multiplier bimonoids A in \mathcal{C} whose multiplication is non-degenerate with respect to some class \mathcal{V} containing I , A and A^2 . In this section we look for further conditions under which A has a canonical quotient L which carries the structure of a coseparable comonoid. Based on this result, in Section 4 we present conditions for suitable A -modules to constitute a monoidal category, whose monoidal structure is lifted from the category of L -bicomodules.

Example 3.1. In abelian categories coequalizers exist, and since every epimorphism is regular, the composite of regular epimorphism is again a regular epimorphism. Thus any

braided monoidal closed abelian category satisfies our assumptions. Several examples of this type will be discussed in Section 6: the category of modules over a commutative ring (and in particular the category of vector spaces over a given field), and the category of vector spaces graded by a given group.

Example 3.2. The symmetric monoidal closed category of complete bornological spaces is not abelian, but it does have coequalizers [18, Section 1.3]. The regular epimorphisms are those $f: X \rightarrow Y$ for which both f itself and the induced function f_* between the bornologies are surjective; it follows easily that the regular epimorphisms are closed under composition.

Example 3.3. Let Hilb be the category whose objects are complex Hilbert spaces and whose morphisms are the continuous linear maps (not required to preserve the inner product). This has finite limits and colimits, and is enriched over abelian groups, but is not an abelian category. It is easy to see that the monomorphisms are the injective maps, and that a monomorphism in Hilb is regular if and only if its image is closed (equivalently, it factorizes as an isomorphism in Hilb followed by an inner-product preserving injection). The epimorphisms are those maps whose codomain is the closure of the image, so that the closure of the image allows any morphism to be factorized as an epimorphism followed by a regular monomorphism. The cokernel $q: K \rightarrow Q$ of a morphism $f: H \rightarrow K$ can be characterized by the following properties:

- q is surjective;
- $q.f$ is zero;
- f corestricts to an epimorphism $H \rightarrow \text{Ker}(q)$.

In particular, the regular epimorphisms are precisely the surjections, and these are clearly closed under composition. Furthermore, both regular monomorphisms and regular epimorphisms are always split: we can use orthogonal projection to construct their left and right inverses, respectively.

If H and K are Hilbert spaces, their tensor product $H \otimes K$ as vector spaces has an inner product, but is not in general complete. If we define $H \hat{\otimes} K$ to be its completion, we obtain a symmetric monoidal structure on Hilb [17, Propositions 2.6.5 and 2.6.12]. A cokernel diagram $H \xrightarrow{f} K \xrightarrow{q} Q$ will be preserved by taking the monoidal product with any Hilbert space L provided that $f \hat{\otimes} 1$ and $q \hat{\otimes} 1$ obey the properties in the characterization of cokernels given above. The second property evidently holds and the first one does because the monoidal product preserves regular epimorphisms (since they are split). In order to verify the third property, note that the monoidal product preserves epimorphisms as well. This in turn follows from the fact that if I is a dense linear subspace of a Hilbert space K , then $I \otimes L$ is a dense linear subspace of $K \otimes L$, for any Hilbert space L . So if $p: H \rightarrow Z$ is an epimorphism, then the image of the equal paths around the diagram

$$\begin{array}{ccc} H \otimes L & \xrightarrow{\quad} & H \hat{\otimes} L \\ p \otimes 1 \downarrow & & \downarrow p \hat{\otimes} 1 \\ Z \otimes L & \xrightarrow{\quad} & Z \hat{\otimes} L \end{array}$$

is dense in $Z \hat{\otimes} L$ so that also the image of the right column is dense for any Hilbert space L . With this preservation of epimorphisms at hand, we see that if f corestricts to an epimorphism $p: H \rightarrow \text{Ker}(q)$ then $f \hat{\otimes} 1$ corestricts to an epimorphism $p \hat{\otimes} 1: H \hat{\otimes} L \rightarrow \text{Ker}(q) \hat{\otimes} L =$

The non-degenerate morphisms in the categories of the above Examples will be investigated in Section 6. In particular, we shall see there that in the category of vector spaces (even if graded by a given group), as well as in Hilb , any morphism which is non-degenerate on either side with respect to the base field — playing the role of the monoidal unit — is non-degenerate on that side with respect to any object of the category in question.

$$\overline{\Pi}_1^R := \text{diagram with a circle labeled } \ell_1 \text{ and a loop} \quad \overline{\Pi}_2^R := \text{diagram with a circle labeled } t_1 \text{ and a loop} \quad (3.1)$$
$$\overline{\Pi}_1^L := \text{diagram with a circle labeled } t_2 \text{ and two external lines} \quad \overline{\Pi}_2^L := \text{diagram with a circle labeled } e_2 \text{ and two external lines} \quad (3.2)$$
$$\sqcap_1^L := \text{diagram with node } e_1 \text{ and two outgoing lines} \quad \sqcap_2^L := \text{diagram with node } t_4 \text{ and two outgoing lines} \quad (3.3)$$
$$\square_1^R := \text{diagram with node } t_3 \text{ and two external lines} \quad \square_2^R := \text{diagram with node } e_2 \text{ and two external lines} \quad (3.4)$$

The symmetries of Corollary 2.12 permute these morphisms: taking the opposite corresponds to interchanging simultaneously the morphisms with and without bar and the labels 1 and 2; taking the coopposite corresponds to interchanging simultaneously the morphisms

with and without bar and the labels L and R ; finally, taking the opposite-coopposite corresponds to interchanging simultaneously the labels R with L and 1 with 2. For example, the opposite of \sqcap_1^L is $\bar{\sqcap}_2^L$, the coopposite is $\bar{\sqcap}_1^R$, while the opposite-coopposite is \sqcap_2^R .

Under the standing assumptions of the section, for a regular weak multiplier bimonoid $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ with underlying object A in \mathcal{C} , consider the coequalizer

$$A^2 \xrightleftharpoons[\bar{\sqcap}_1^R \cdot c^{-1}]{\sqcap_1^L} A \xrightarrow{p} L \quad (3.5)$$

in \mathcal{C} . We will refer to the object L as the *base object* of A ; note that it is unique up to isomorphism. By the first equality of A.10 and A.11, and by (2.1) and non-degeneracy of the multiplication, (3.5) determines a unique \mathbb{M} -morphism $n: L \rightarrowtail A$ with components occurring in the diagrams

$$\begin{array}{ccc} A^3 \xrightleftharpoons[\bar{\sqcap}_1^R \cdot 1 \cdot c^{-1}]{\sqcap_1^L \cdot 1} A^2 & \xrightarrow{p_1} & LA \\ & \searrow \sqcap_1^L & \downarrow n_1 \\ & & A \end{array} \quad \begin{array}{ccc} A^3 \xrightleftharpoons[\bar{\sqcap}_1^R \cdot 1 \cdot c^{-1}]{1 \cdot \sqcap_1^L} A^2 & \xrightarrow{1p} & AL \\ & \searrow \sqcap_2^L & \downarrow n_2 \\ & & A. \end{array} \quad (3.6)$$

Recall that if the multiplication m of A is non-degenerate with respect to some class \mathcal{V} then n_1 is non-degenerate on the right with respect to \mathcal{V} if and only if n_2 is non-degenerate on the left with respect to \mathcal{V} .

Theorem 3.4. *Let \mathcal{C} be a braided monoidal category in which coequalizers exist and are preserved by the monoidal product, and the composite of regular epimorphisms is a regular epimorphism. Let $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ be a regular weak multiplier bimonoid in \mathcal{C} with underlying object A such that its multiplication in Lemma 2.7 is a regular epimorphism and non-degenerate with respect to some class \mathcal{V} containing I , A , A^2 and the object L from (3.5). Assume further that the morphism n_1 of (3.6) is non-degenerate on the right with respect to \mathcal{V} . Then the following hold.*

- (1) *There is an associative multiplication $\mu: L^2 \rightarrow L$ with respect to which n_1 is an associative action.*
- (2) *There is a coassociative comultiplication $\delta: L \rightarrow L^2$ rendering commutative*

$$\begin{array}{ccccc} A^2 & \xrightarrow{m} & A & \xrightarrow{p} & L \\ t_1 \downarrow & & & & \downarrow \delta \\ A^2 & \xrightarrow{pp} & L^2 & & \end{array}$$

- (3) *The equality $L \xrightarrow{\delta} L^2 \xrightarrow{\mu} L = 1$ holds.*
- (4) *The comultiplication δ is a morphism of L -bimodules. That is, the following diagram commutes.*

$$\begin{array}{ccc} L^2 & \xrightarrow{1\delta} & L^3 \\ \delta 1 \downarrow & \searrow \mu & \downarrow \mu 1 \\ L^3 & \xrightarrow{1\mu} & L^2 \end{array}$$

(5) The comultiplication δ admits a counit ε satisfying $\varepsilon.p = j$.

In particular, L carries the structure of a coseparable comonoid.

Proof. (1) The top row of

$$\begin{array}{ccccc}
 LA^2 & \xrightleftharpoons[1\overline{\Pi}_1^R.1c^{-1}]{1\overline{\Pi}_1^L} & LA & \xrightarrow{1p} & L^2 \\
 & & \downarrow n_1 & & \downarrow \mu \\
 & & A & \xrightarrow{p} & L
 \end{array}$$

is a coequalizer and the left-bottom path coequalizes the parallel arrows of the top row by

and the fact that $\overline{\Pi}_1^L = n_1.p1$ and $p11: A^3 \rightarrow LA^2$ is an epimorphism. This proves the existence of a unique morphism μ as in the diagram. It also renders commutative the diagrams

$$\begin{array}{ccc}
 A^2 & \xrightarrow{pp} & L^2 \\
 \overline{\Pi}_1^L \downarrow & & \downarrow \mu \\
 A & \xrightarrow{p} & L
 \end{array}
 \quad
 \begin{array}{ccc}
 A^2 & \xrightarrow{pp} & L^2 \\
 \overline{\Pi}_1^R.c^{-1} \downarrow & & \downarrow \mu \\
 A & \xrightarrow{p} & L.
 \end{array}$$

The stated associativity properties follow by A.10 together with the fact that $ppp: A^3 \rightarrow L^3$ and $pp1: A^3 \rightarrow L^2A$ are epimorphisms.

(2) The top row of the diagram of part (2) is a regular epimorphism by assumption. It follows by A.22 — applied together with the non-degeneracy conditions on the multiplication and n_1 — that the left-bottom path coequalizes those morphisms whose coequalizer is in the top row. Thus the desired (unique) morphism δ exists by universality.

In the diagrams

the top-left regions commute by A.1, and by (2.3) (short fusion equation) for the weakly counital fusion morphism (t_1, e_1, j) . All other regions commute by the construction of δ .

The top rows are equal epimorphisms by the associativity of m . Since the left verticals are equal by Axiom I for t_1 , this proves the coassociativity of δ .

(3) In the diagram

$$\begin{array}{ccccc} A^2 & \xrightarrow{m} & A & \xrightarrow{p} & L \\ t_1 \downarrow & & & & \downarrow \delta \\ A^2 & \xrightarrow{pp} & & & L^2 \\ \sqcap_1^L \downarrow & & & & \downarrow \mu \\ A & \xrightarrow{p} & & & L \end{array}$$

both regions commute by the constructions of δ and μ , respectively. By A.7 the left-bottom path and the top row are equal epimorphisms, which proves that the right vertical is the identity morphism.

(4) In the second of the diagrams

$$\begin{array}{ccc} A^3 \xrightarrow{1m} A^2 \xrightarrow{pp} L^2 & & A^3 \xrightarrow{1m} A^2 \xrightarrow{pp} L^2 \\ t_1 \downarrow & & \sqcap_1^L \downarrow \\ A^3 \xrightarrow{ppp} L^3 & & A^2 \xrightarrow{m} A \xrightarrow{p} L \\ \sqcap_1^L \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \end{array} \quad \begin{array}{ccc} A^3 \xrightarrow{1m} A^2 \xrightarrow{pp} L^2 & & A^3 \xrightarrow{1m} A^2 \xrightarrow{pp} L^2 \\ \sqcap_1^L \downarrow & & \sqcap_1^L \downarrow \\ A^2 \xrightarrow{m} A \xrightarrow{p} L & & A^2 \xrightarrow{m} A \xrightarrow{p} L \\ t_1 \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \end{array}$$

the top-left region commutes by A.1 and all other regions commute by the construction of μ or δ . The top rows are epimorphisms and the left verticals are equal by the first identity in A.5. Hence the right verticals are equal, which proves the left L -linearity of δ . Similarly, in the first of the diagrams

$$\begin{array}{ccc} A^4 \xrightarrow{mm} A^2 \xrightarrow{pp} L^2 & & A^4 \xrightarrow{mm} A^2 \xrightarrow{pp} L^2 \\ m1 \downarrow & & 11m \downarrow \\ A^3 & & A^3 \\ \sqcap_1^L \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{m} A \xrightarrow{p} A & & A^3 \xrightarrow{ppp} L^3 \\ t_1 \downarrow & & 1 \sqcap_1^L \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \end{array} \quad \begin{array}{ccc} A^4 \xrightarrow{mm} A^2 \xrightarrow{pp} L^2 & & A^4 \xrightarrow{mm} A^2 \xrightarrow{pp} L^2 \\ m1 \downarrow & & 11m \downarrow \\ A^3 & & A^3 \\ \sqcap_1^L \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{m} A \xrightarrow{p} A & & A^3 \xrightarrow{ppp} L^3 \\ t_1 \downarrow & & 1 \sqcap_1^L \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \end{array}$$

the top-left region commutes by A.1 and all other regions commute by the construction of μ or δ . The top rows are epimorphisms and the left-bottom paths are equal by A.24, applied together with the non-degeneracy conditions on n_1 and n_2 , which proves the right L -linearity of δ .

(5) The morphism j evidently coequalizes the parallel arrows of (3.5), which proves the existence of ε as in the claim. The diagrams

$$\begin{array}{ccc} A^2 \xrightarrow{m} A \xrightarrow{p} L & & A^2 \xrightarrow{m} A \xrightarrow{p} L \\ t_1 \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \\ j1 \downarrow & & 1j \downarrow \\ A \xrightarrow{p} L & & A \xrightarrow{p} L \end{array} \quad \begin{array}{ccc} A^2 \xrightarrow{m} A \xrightarrow{p} L & & A^2 \xrightarrow{m} A \xrightarrow{p} L \\ t_1 \downarrow & & t_1 \downarrow \\ A^2 \xrightarrow{pp} L^2 & & A^2 \xrightarrow{pp} L^2 \\ j1 \downarrow & & 1j \downarrow \\ A \xrightarrow{p} L & & A \xrightarrow{p} L \end{array}$$

commute by definition of δ and ε . The left-bottom composite in the first diagram is equal to $p.m$ by (2.14), while in the second diagram this is true by the first equality in A.12 and the non-degeneracy of n_2 . This proves that their right verticals are identity morphisms; that is, ε is the counit of δ . \square

Remark 3.5. Symmetrically to the construction of the comultiplication δ in the proof of Theorem 3.4, we can define it as the unique morphism rendering commutative

$$\begin{array}{ccc} A^2 & \xrightarrow{m} & A \xrightarrow{p} L \\ c \downarrow & & \downarrow \delta \\ A^2 & & L \\ t_4 \downarrow & & \downarrow \gamma \\ A^2 & \xrightarrow{pp} & L^2 \end{array} \quad (3.7)$$

It follows by A.19, applied together with the non-degeneracy conditions on n_1 and n_2 , that this yields the same morphism δ .

Remark 3.6. Let us recall from [12] that in a coseparable comonoid (L, δ, ε) , the bicomodule section μ of δ is a *firm* multiplication in the sense of [21]. That is, it is an associative multiplication such that

$$L^3 \xrightleftharpoons[1\mu]{\mu 1} L^2 \xrightarrow{\mu} L$$

is a coequalizer. Consequently [11, 3], left L -comodules can be identified with firm left L -modules; that is, with associative L -actions $\xi : LX \rightarrow X$ such that

$$L^2X \xrightleftharpoons[1\xi]{\mu 1} LX \xrightarrow{\xi} X$$

is a coequalizer. Explicitly, an L -coaction $\tau : X \rightarrow LX$ determines a firm L -action

$$\xi := LX \xrightarrow{1\tau} L^2X \xrightarrow{\mu 1} LX \xrightarrow{\varepsilon 1} X$$

and, conversely, a firm L -action $\xi : LX \rightarrow X$ determines a unique L -coaction τ such that

$$\begin{array}{ccc} LX & \xrightarrow{\xi} & X \\ \delta 1 \downarrow & & \downarrow \tau \\ L^2X & \xrightarrow{1\xi} & LX \end{array}$$

commutes. There is a symmetric correspondence between right L -comodules and firm right L -modules.

Summarizing, an L -bicomodule is the same as a left and right firm L -bimodule. To avoid the use of unnecessary multiple terminology, in this paper we will only speak about L -comodules (but keeping the above correspondence in mind).

Remark 3.7. Consider a regular weak multiplier bialgebra A over a field; this includes the assumptions that the multiplication is surjective and non-degenerate. In [4] the base object of A was defined as the image of the map \square^L from A to its multiplier algebra. We claim that — whenever the comultiplication of A is *left full* in the sense of [4, Theorem 3.13] — this gives the same vector space as the coequalizer L in (3.5).

In this case we can write $\sqcap_1^L(a \otimes b) = \sqcap^L(a)b$ and $\sqcap_1^R(a \otimes b) = \sqcap^R(a)b$ for all $a, b \in A$, in terms of maps \sqcap^L and \sqcap^R from A to the multiplier algebra of A (see (6.3)). By [4, Lemma 3.8] the map \sqcap^L coequalizes the parallel morphisms of (3.5). Hence there is a unique epimorphism $f: L \rightarrow \text{Im}(\sqcap^L)$ such that $\sqcap^L(a) = f(p(a))$ for all $a \in A$. In order to see that f is injective as well, we need to show that $f(p(a)) = 0$ for some a if and only if $p(a) = 0$. Equivalently, $\sqcap^L(a) = 0$ if and only if a belongs to the image of $\sqcap_1^R \cdot c - \sqcap_1^L$.

By [4, Theorem 4.7 and Lemma 4.8], the non-unital algebra $\text{Im}(\sqcap^R)$ possesses local units; and by (a symmetric variant of) [4, Proposition 5.2] A is a firm module over it (in the sense of [21]). Hence for any element a of A there is an associated element b of A such that $\sqcap^R(b)a = a$. Choose a such that $\sqcap^L(a) = 0$. Then

$$a = \sqcap^R(b)a = \sqcap^R(b)a - \sqcap^L(a)b.$$

So we conclude that in this situation the canonical surjection $A \rightarrow \text{Im}(\sqcap^L)$ is the coequalizer in (3.5).

In fact, as we shall see in Section 6.3, in the category of vector spaces the morphism n_1 of (3.6) is non-degenerate on the right (with respect to the base field, equivalently, with respect to any vector space) if and only if L in (3.5) is isomorphic to the image of the map \sqcap^L . As the above considerations show, these properties hold whenever the comultiplication of A is left full.

In this sense Theorem 3.4 gives a new insight also to regular weak multiplier bialgebras over fields: It says that the assumption about the *fullness of the comultiplication* in [4, Theorem 4.7] can be replaced by the *non-degeneracy of n_1* .

4. THE MONOIDAL CATEGORY OF MODULES

If $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ is a regular weak multiplier bimonoid in a braided monoidal category \mathcal{C} such that all assumptions of Theorem 3.4 hold, then Theorem 3.4 says that the base object L carries the structure of a coseparable comonoid. Consequently, whenever idempotent morphisms in \mathcal{C} split, the category of L -bicomodules is monoidal. The monoidal unit is L , with both coactions given by the comultiplication δ . The monoidal product of L -bicomodules V and W — with left coactions denoted by τ and right coactions denoted by $\bar{\tau}$ — is the object $V \circ W$ occurring in the splitting

$$VW \xrightarrow{\hat{s}} V \circ W \xrightarrow{\check{s}} VW$$

of the idempotent morphism

$$s := VW \xrightarrow{\bar{\tau}\tau} VL^2W \xrightarrow{1\mu 1} VLW \xrightarrow{1\varepsilon 1} VW. \quad (4.1)$$

(It can be regarded as the usual L -comodule tensor product, equivalently, as the module tensor product of the firm L -modules V and W , see Remark 3.6.)

The aim of this section is to see — under the hypotheses of Theorem 3.4, and assuming that idempotent morphisms in \mathcal{C} split — what else is needed for the category of modules (in an appropriate sense, see below) over a regular weak multiplier bimonoid to be monoidal via the lifting of this monoidal structure on the category of L -bicomodules.

Throughout this section, we assume that \mathcal{C} is a braided monoidal category in which coequalizers exist and are preserved by the monoidal product, and that the composite of regular epimorphisms is a regular epimorphism. We further assume that $(A, t_1, t_2, t_3, t_4, e_1, e_2, j)$

is a regular weak multiplier bimonoid in \mathcal{C} whose multiplication $m = j1.t_1$ is a regular epimorphism and non-degenerate with respect to some class \mathcal{Y} containing A , the unit object I , and the object L from (3.5), and closed under the monoidal product. Finally, the morphism n_1 of (3.6) is assumed to be non-degenerate on the right with respect to \mathcal{Y} ; equivalently, n_2 of (3.6) is assumed to be non-degenerate on the left with respect to \mathcal{Y} .

Note that, without any loss of generality, we may assume that \mathcal{Y} is closed under retracts. Indeed, if $i: Z \rightarrow X$ is a monomorphism preserved by the functor $(-)_Q$ for any object Q (as happens for example if i has a left inverse), and some morphism $v: QV \rightarrow W$ is non-degenerate on the right with respect to X , then it is non-degenerate on the right with respect to Z as well: from the composite

$$PV \xrightarrow{f1} ZQV \xrightarrow{1v} ZW$$

we can uniquely recover any $f: P \rightarrow ZQ$ by post-composing with $i1$; applying the non-degeneracy with respect to X to recover $i1.f$; and then using the fact that $i1: ZQ \rightarrow XQ$ is a monomorphism.

For some results in the section, we will also need to assume that idempotent morphisms in \mathcal{C} split. This happens, for example, if any morphism of \mathcal{C} admits an epi-mono factorization, or if \mathcal{C} has coequalizers or equalizers.

Example 4.1. In an abelian category any morphism f has an epi-mono factorization through the image of f . This includes in particular some examples from Section 6: the category of modules over a commutative ring (so in particular the category of vector spaces over a given field) and the category of group-graded vector spaces.

Example 4.2. In the (non-abelian) category of complete bornological vector spaces we can factorize any morphism $f: X \rightarrow Y$ through the image $\text{Im}(f)$, computed in the category of vector spaces. We may equip $\text{Im}(f) \cong X/\text{Ker}(f)$ with the quotient bornology (which can be different, however, from the subspace bornology of $\text{Im}(f) \subseteq Y$). Since it is complete, this gives a factorization of f as a composite of a regular epimorphism $X \rightarrow \text{Im}(f)$ (see Example 3.2) and a monomorphism $\text{Im}(f) \rightarrow Y$.

Example 4.3. The category Hilb is finitely complete and cocomplete, so idempotents split.

Proposition 4.4. *Under the standing assumptions of the section, the object A carries the structure of a bicomodule over the comonoid L in Theorem 3.4.*

Proof. By A.13 and the non-degeneracy conditions on the multiplication and n_2 , the left-bottom path of

$$\begin{array}{ccc} A^2 & \xrightarrow{m} & A \\ t_1 \downarrow & & \downarrow \tau \\ A^2 & \xrightarrow{p1} & LA \end{array}$$

coequalizes any pair of morphisms that the top row does. So we can use the universality of the coequalizer in the top row to construct a left L -coaction τ on A . By the constructions of τ and δ , by the short fusion equation (2.3) on t_1 , by A.1, and by functoriality of the

monoidal product, both diagrams

$$\begin{array}{ccc}
 A^3 & \xrightarrow{m1} & A^2 \xrightarrow{m} A \\
 \downarrow l_{t_1} & & \downarrow t_1 \quad \downarrow \tau \\
 A^3 & & \\
 \downarrow c1 & & \\
 A^2 & & \\
 \downarrow l_{t_1} & & \\
 A^3 & & \\
 \downarrow c^{-1}1 & & \\
 A^3 & \xrightarrow{m1} & A^2 \xrightarrow{p1} LA \\
 \downarrow t_1 1 & & \downarrow \delta 1 \\
 A^3 & \xrightarrow{pp1} & L^2 A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^3 & \xrightarrow{1m} & A^2 \xrightarrow{m} A \\
 \downarrow t_1 1 & & \downarrow t_1 \quad \downarrow \tau \\
 A^3 & & \\
 \downarrow 1t_1 & & \\
 A^3 & \xrightarrow{1m} & A^2 \xrightarrow{p1} LA \\
 \downarrow 1t_1 & & \downarrow 1\tau \\
 A^3 & \xrightarrow{1p1} & ALA \xrightarrow{p11} L^2 A
 \end{array}$$

commute. Since their top rows are equal epimorphisms by the associativity of m , and their left verticals are equal by Axiom **I**, we conclude that τ is coassociative. Also

$$\begin{array}{ccc}
 A^2 & \xrightarrow{m} & A \\
 \downarrow t_1 & & \downarrow \tau \\
 A^2 & \xrightarrow{p1} & LA \\
 \downarrow j1 & & \downarrow \varepsilon 1 \\
 A & \xlongequal{\quad} & A
 \end{array}$$

commutes thanks to the constructions of τ and ε . By (2.14) the left vertical is equal to the epimorphism in the top row, proving that the right vertical is the identity morphism. That is to say, τ is also counital.

Similarly, by the coopposite of the first equality of A.3 and by the non-degeneracy of n_1 on the right with respect to A , the left-bottom path of

$$\begin{array}{ccc}
 A^2 & \xrightarrow{m} & A \\
 \downarrow c & & \downarrow \bar{\tau} \\
 A^2 & & \\
 \downarrow t_4 & & \downarrow \\
 A^2 & \xrightarrow{1p} & AL
 \end{array}$$

coequalizes any pair of morphisms that the top row does. So we can use the universality of the coequalizer in the top row to construct a right L -coaction $\bar{\tau}$ on A . By the construction of $\bar{\tau}$, by (3.7), the short fusion equation (2.3) for t_4 , naturality of the braiding, A.1, and by the

functoriality of the monoidal product, both diagrams

$$\begin{array}{ccc}
 A^3 & \xrightarrow{m1} & A^2 \xrightarrow{m} A \\
 c_{A^2,A} \downarrow & & \downarrow c \\
 A^3 & \xrightarrow{1m} & A^2 \\
 1c \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 t_4 1 \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 1c^{-1} \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 t_4 1 \downarrow & & \downarrow t_4 \\
 A^3 & \xrightarrow{1m} A^2 \xrightarrow{1p} AL & \\
 1c \downarrow & & \downarrow 1\delta \\
 A^3 & & AL \\
 1t_4 \downarrow & & \downarrow 1\delta \\
 A^3 & \xrightarrow{1pp} AL^2 &
 \end{array}
 \quad
 \begin{array}{ccc}
 A^3 & \xrightarrow{1m} & A^2 \xrightarrow{m} A \\
 c_{A,A^2} \downarrow & & \downarrow c \\
 A^3 & \xrightarrow{m1} & A^2 \\
 1c^{-1} \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 t_4 1 \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 1c \downarrow & & \downarrow t_4 \\
 A^3 & \xrightarrow{m1} A^2 \xrightarrow{1p} AL & \\
 c1 \downarrow & & \downarrow \bar{\tau}1 \\
 A^3 & & AL \\
 t_4 1 \downarrow & & \downarrow \bar{\tau}1 \\
 A^3 & \xrightarrow{1p1} ALA \xrightarrow{11p} AL^2 &
 \end{array}$$

commute. The top rows are equal epimorphisms by the associativity of m and the left verticals are equal by the coopposite of Axiom I (the fusion equation for t_4), so we conclude that $\bar{\tau}$ is coassociative. The diagram

$$\begin{array}{ccc}
 A^2 & \xrightarrow{m} & A \\
 c \downarrow & & \downarrow \bar{\tau} \\
 A^2 & & A \\
 t_4 \downarrow & & \downarrow \bar{\tau} \\
 A^2 & \xrightarrow{1p} & AL \\
 1j \downarrow & & \downarrow 1\epsilon \\
 A & \xrightarrow{=} & A
 \end{array}$$

is commutative by the constructions of $\bar{\tau}$ and ϵ . By (2.17), the left vertical is equal to the epimorphism in the top row, proving that the right vertical is the identity morphism. Thus $\bar{\tau}$ is counital. Finally, by the constructions of τ and $\bar{\tau}$, by A.1 and by the functoriality of the monoidal product, the diagrams

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1m} & A^2 \xrightarrow{m} A \\
 c1 \downarrow & & \downarrow c \\
 A^3 & & A^2 \\
 t_4 1 \downarrow & & \downarrow t_4 \\
 A^3 & & A^2 \\
 1c \downarrow & & \downarrow t_4 \\
 A^3 & \xrightarrow{m1} A^2 \xrightarrow{1p} AL & \\
 t_1 1 \downarrow & & \downarrow \tau 1 \\
 A^3 & \xrightarrow{p11} LA^2 \xrightarrow{11p} LAL &
 \end{array}
 \quad
 \begin{array}{ccc}
 A^3 & \xrightarrow{1m} & A^2 \xrightarrow{m} A \\
 t_1 1 \downarrow & & \downarrow t_1 \\
 A^3 & & A^2 \\
 1c \downarrow & & \downarrow t_1 \\
 A^3 & \xrightarrow{1m} A^2 \xrightarrow{p1} LA & \\
 1t_4 \downarrow & & \downarrow 1\bar{\tau} \\
 A^3 & \xrightarrow{11p} A^2L \xrightarrow{p11} LAL &
 \end{array}$$

commute. Their top rows are equal epimorphisms by the associativity of m and their left-bottom paths are equal by A.23 applied together with the non-degeneracy conditions on n_1 and n_2 . This proves that the left and right coactions τ and $\bar{\tau}$ on A commute. \square

Definition 4.5. By a *module* over a semigroup (M, m) we mean an object V in \mathcal{V} together with a morphism $v: MV \rightarrow V$ — called the *action* — subject to the following conditions.

- v is associative; that is, the following diagram commutes.

$$\begin{array}{ccc} M^2V & \xrightarrow{1v} & MV \\ m1 \downarrow & & \downarrow v \\ MV & \xrightarrow{v} & V \end{array}$$

- v is a regular epimorphism.
- v is non-degenerate on the left with respect to the class \mathcal{V} .

A *morphism of modules* is a morphism $f: V \rightarrow V'$ which is compatible with the actions in the sense of the commutative diagram

$$\begin{array}{ccc} MV & \xrightarrow{1f} & MV' \\ v \downarrow & & \downarrow v' \\ V & \xrightarrow{f} & V' \end{array}$$

Theorem 4.6. Under the standing assumptions of the section, for any module $v: AV \rightarrow V$ of the semigroup A , the object V admits the structure of a bicomodule over the comonoid L in Theorem 3.4. Any morphism of A -modules is a bicomodule morphism with respect to these L -coactions. Hence there is a functor U from the category of A -modules to the category of L -bicomodules which acts on the morphisms as the identity map.

Proof. In the diagram

$$\begin{array}{ccccc} A^3 & \xleftarrow{11m} & A^4 & \xrightarrow{11m} & A^3 \\ t_3 1 \downarrow & & \downarrow c^{-1} 11 & & \downarrow c^{-1} 1 \\ A^3 & & A^4 & & A^3 \\ c^{-1} 1 \downarrow & \nearrow \sqcap_2^L 1 & \downarrow m 11 & & \downarrow m 1 \\ A^3 & \xrightarrow{1 \sqcap_1^L} & A^2 & \xleftarrow{n_2 1} & ALA & \xleftarrow{1 \tau} & A^2 & \xrightarrow{=} & A^2 \end{array}$$

the large region on the left commutes by A.13. All other regions commute by the construction of τ and functoriality of the monoidal product. Since those of the top row are equal epimorphisms, we deduce the equality of the left-bottom and right-bottom paths. Using it in the third equality, together with the associativity of v in the first and the penultimate equalities, with (2.1) in the second equality, and with (2.2) in the last equality, we obtain

By the non-degeneracy of v , m and n_2 on the left with respect to \mathcal{V} , this proves that the left-bottom path of

$$\begin{array}{ccc} AV & \xrightarrow{v} & V \\ \tau 1 \downarrow & & \downarrow \tau \\ LAV & \xrightarrow{1v} & LV \end{array} \quad (4.2)$$

coequalizes any pair of morphisms that the top row does. Thus we can use the universality of the coequalizer in the top row of (4.2) to define a left L -coaction τ on V .

Similarly, using the fact that $m11: A^4 \rightarrow A^3$ is epi, it follows from the construction of the right L -coaction $\bar{\tau}$ on A , equation (2.2) and the opposite of the second equality of A.14 that

$$\begin{array}{ccccc} A^3 & \xrightarrow{1m} & A^2 & \xrightarrow{\bar{\tau}1} & ALA \\ 1c \downarrow & & & & \downarrow 1c^{-1} \\ A^3 & & & & A^2L \\ 1t_3 \downarrow & & & & \downarrow 1n_2 \\ A^3 & \xrightarrow{c^{-1}1} & A^3 & \xrightarrow{\square_1^R 1} & A^2 \end{array}$$

commutes. With its help one derives

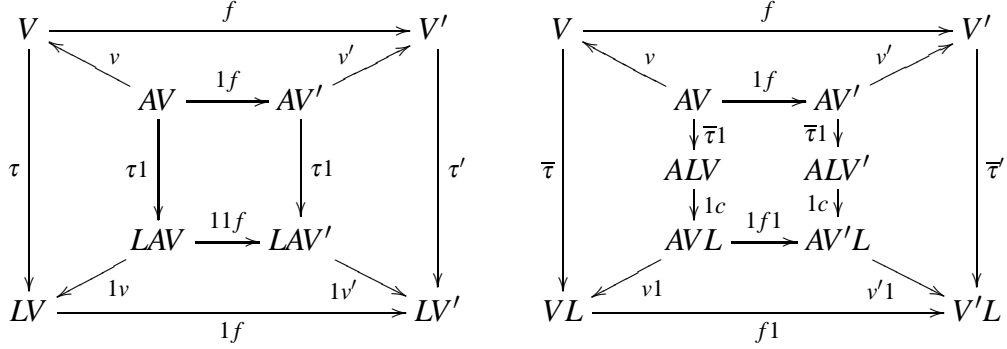
By the non-degeneracy of m , v and n_2 on the left with respect to \mathcal{V} , this implies that the left-bottom path of

$$\begin{array}{ccc} AV & \xrightarrow{v} & V \\ \bar{\tau}1 \downarrow & & \downarrow \bar{\tau} \\ ALV & & VL \\ 1c \downarrow & & \downarrow \\ AVL & \xrightarrow{v1} & VL \end{array} \quad (4.3)$$

coequalizes any pair of morphisms that the top row does. Thus we can use the universality of the coequalizer in the top row of (4.3) to define a right L -coaction $\bar{\tau}$ on V .

The left L -coaction τ on V is coassociative and counital by the coassociativity and the counitality of the left L -coaction τ on A ; the right L -coaction $\bar{\tau}$ on V is coassociative and counital by the coassociativity and the counitality of the right L -coaction $\bar{\tau}$ on A ; and the coactions τ and $\bar{\tau}$ on V commute since the coactions τ and $\bar{\tau}$ on A do.

For any morphism $f: V \rightarrow V'$ of A -modules, in the diagrams



the regions at the middle commute by the functoriality of the monoidal product and the naturality of the braiding. The regions on the left and on the right commute by the constructions of the coactions τ and $\bar{\tau}$. The upper and lower regions commute because f is a morphism of A -modules. Since $v: AV \rightarrow V$ is epi, this proves that f is a morphism of left and right L -comodules. \square

Our final aim is to lift the monoidal structure on the category of L -bicomodules through the functor U in Theorem 4.6 to give a monoidal structure on the category of A -modules.

Proposition 4.7. *Under the standing assumptions of the section, the base object L is an A -module via the action*

$$AL \xrightarrow{n_2} A \xrightarrow{p} L,$$

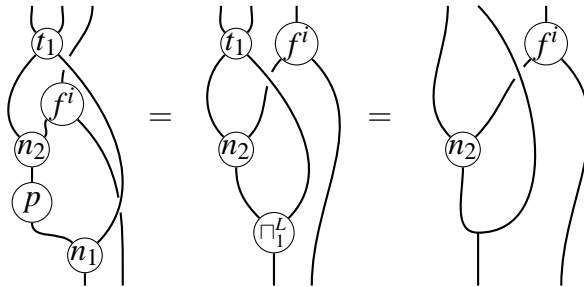
where p and n_2 are defined as in (3.5) and (3.6), respectively.

Proof. The object L belongs to the class \mathcal{Y} by assumption. The region at the right of

$$\begin{array}{ccccc} A^2 & \xrightarrow{m} & A & & \\ 1p \downarrow & \searrow \lrcorner_2^L & \downarrow p & & \\ AL & \xrightarrow{n_2} & A & \xrightarrow{p} & L \end{array} \quad (4.4)$$

commutes by the second identity of A.12 applied together with the non-degeneracy of n_2 on the right. The top-right path is a composite of regular epimorphisms, thus it is a regular epimorphism, hence so is the left-bottom path. Since the left column is an epimorphism, this proves that the bottom row is a regular epimorphism. It follows immediately by commutativity of the diagram of (4.4) and the associativity of m in the top row that the bottom row of (4.4) is an associative action.

It remains to see the non-degeneracy of the stated action on the left with respect to \mathcal{Y} . If $p1.n_21$ coequalizes $1f^1$ and $1f^2$ for some morphisms f^1 and f^2 to LY where $Y \in \mathcal{Y}$, then the morphism



does not depend on $i \in \{1, 2\}$. The second equality follows by A.20 using that $11p: A^3 \rightarrow A^2L$ is epi. By the non-degeneracy of m on the right and n_2 on the left with respect to \mathcal{Y} , this implies $f^1 = f^2$ hence non-degeneracy of the A -action on L on the left with respect to \mathcal{Y} . \square

Lemma 4.8. *Under the standing assumptions of the section, for any modules $v: AV \rightarrow V$ and $w: AW \rightarrow W$ over the semigroup A , the following diagrams commute.*

$$\begin{array}{ccc} A^2VW \xrightarrow{1c1} AVAW \xrightarrow{vw} VW & & A^2VW \xrightarrow{11s} A^2VW \xrightarrow{1c1} AVAW \\ e_111 \downarrow & & e_211 \downarrow \\ A^2VW \xrightarrow{1c1} AVAW \xrightarrow{vw} VW & & A^2VW \xrightarrow{1c1} AVAW \xrightarrow{vw} VW \end{array}$$

where s is the morphism (4.1).

Proof. The diagram

$$\begin{array}{ccccc} A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW \\ \bar{\tau}\tau11 \downarrow & & \downarrow \bar{\tau}1\tau1 & & \downarrow \bar{\tau}\tau \\ AL^2AVW & \xrightarrow{11c_{LA,V}1} & ALVLAW & & \\ \downarrow 1\mu111 & \searrow 1c_{L^2A,V}1 & \downarrow 1c111 & & \downarrow \bar{\tau}\tau \\ & & AVL^2AW & \xrightarrow{v11w} & VL^2W \\ & & \downarrow 11\mu11 & & \downarrow 1\mu1 \\ ALAVW & \xrightarrow{1c_{LA,V}1} & AVLAW & & VLW \\ \downarrow 1\varepsilon111 & & \downarrow 11\varepsilon11 & & \downarrow 1\varepsilon1 \\ A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW \end{array} \quad (4.5)$$

is commutative by the constructions of the L -coactions on V and W , and naturality and coherence of the braiding. Since $mm: A^4 \rightarrow A^2$ is epi, commutativity of

$$\begin{array}{ccccc} A^2 & \xleftarrow{mm} & A^4 & \xrightarrow{mm} & A^2 \\ \downarrow \bar{\tau}\tau & & \downarrow c11 & & \downarrow e_1 \\ & & A^4 & & \\ & & \downarrow t_4t_1 & & \\ & & A^4 & \xrightarrow{1\cap_1^L1} & A^3 \xrightarrow{1j1} A^2 \\ & & \downarrow 1pp1 & & \downarrow 1p1 \\ AL^2A & \xrightarrow{1\mu1} & ALA & \xrightarrow{1\varepsilon1} & A^2 \end{array} \quad (4.6)$$

implies that the left-bottom path of (4.5) is equal to $vw.1c1.e_111$. Hence so is the top-right path, proving commutativity of the first diagram of the claim. In (4.6) the region on the left commutes by the constructions of the coactions $\bar{\tau}$ and τ , the middle region at the bottom commutes by the construction of μ , the bottom-right region commutes by the construction of ε , and the top-right region commutes by the second equality of A.3.

In order to prove commutativity of the second diagram of the claim, note that $b^{(2)} := A^2VW \xrightarrow{-1c1} AVAW \xrightarrow{-vw} VW$ is an associative action for the multiplication $m^{(2)} := A^4 \xrightarrow{-1c1} A^4 \xrightarrow{-mm} A^2$. Hence the bottom regions of the diagram

$$\begin{array}{ccccc}
 A^2VW & \xleftarrow{11b^{(2)}} & A^4VW & \xrightarrow{11b^{(2)}} & A^2VW \\
 \downarrow e_2 11 & & \swarrow e_2 1111 & \searrow 11e_1 11 & \downarrow 11s \\
 & & A^4VW & & A^4VW \\
 & \swarrow 11b^{(2)} & \searrow m^{(2)} 11 & \swarrow m^{(2)} 11 & \searrow 11b^{(2)} \\
 A^2VW & & A^2VW & & A^2VW \\
 \parallel & & \downarrow b^{(2)} & & \parallel \\
 A^2VW & \xrightarrow{b^{(2)}} & VW & \xleftarrow{b^{(2)}} & A^2VW
 \end{array}$$

commute. The region at the middle commutes by (2.18), the triangle-shaped region at the top-left commutes by functoriality of the monoidal product and the triangle-shaped region at the top-right commutes by commutativity of the first diagram of the claim. Since the morphisms of the top row are equal epimorphisms, this completes the proof. \square

Lemma 4.9. *Under the standing assumptions of the section, for any modules $v: AV \rightarrow V$ and $w: AW \rightarrow W$ over the semigroup A , the object VW admits an associative action $b^0: AVW \rightarrow VW$ (which may not obey the non-degeneracy conditions on a module), which renders commutative the diagrams*

$$\begin{array}{ccc}
 AVW & \xrightarrow{1s} & AVW \\
 b^0 \downarrow & \searrow b^0 & \downarrow b^0 \\
 VW & \xrightarrow{s} & VW
 \end{array}
 \quad
 \begin{array}{ccccc}
 A^3VW & \xrightarrow{d_2 11} & A^2VW & \xrightarrow{1c1} & AVAW \\
 11b^0 \downarrow & & \downarrow 11b^0 & & \downarrow vw \\
 A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW
 \end{array}$$

in which s is the morphism (4.1) and $d_2 := A^3 \xrightarrow{c^{-1}} A^3 \xrightarrow{1t_2} A^3 \xrightarrow{c1} A^3 \xrightarrow{1m} A^2$, of A.9.

Proof. For the morphism $d_1 := A^3 \xrightarrow{1c^{-1}} A^3 \xrightarrow{t_1 1} A^3 \xrightarrow{1c} A^3 \xrightarrow{m1} A^2$ of A.9,

$$\begin{array}{ccccccc}
 \begin{array}{c} \text{Diagram 1: } A^3 \text{ with } d_1, v, w \end{array} & \stackrel{(a)}{=} & \begin{array}{c} \text{Diagram 2: } A^3 \text{ with } d_1, v, w \end{array} & \stackrel{\text{A.9}}{=} & \begin{array}{c} \text{Diagram 3: } A^3 \text{ with } t_2, v, w \end{array} & \stackrel{(a)}{=} & \begin{array}{c} \text{Diagram 4: } A^3 \text{ with } t_2, v, w \end{array}
 \end{array}$$

where we are now using (a) for associativity of the actions. By the non-degeneracy of v on the left with respect to \mathcal{V} , this implies that the morphism in the left-bottom path of

$$\begin{array}{ccccc}
 A^3VW & \xrightarrow{11c1} & A^2VAW & \xrightarrow{1vw} & AVW \\
 d_1 11 \downarrow & & & & \downarrow b^0 \\
 A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW
 \end{array}$$

coequalizes morphisms $1f$ and $1g$ whenever $vw.1c1$ coequalizes f and g . The top row is the image of the coequalizer $vw.1c1$ under the functor $A(-)$ hence it is a coequalizer; we can use its universality to define b^0 . It is associative by the associativity of d_1 : see the third equality of A.9.

Concerning the commutativity of the diagrams of the claim, we use again the associative action $b^{(2)} := A^2VW \dashv_{1c1} \Rightarrow AVW \dashv_{vw} \Rightarrow VW$ for the multiplication $m^{(2)} := A^4 \dashv_{1c1} \Rightarrow A^4 \dashv_{mm} \Rightarrow A^2$. Observe that the diagrams

$$\begin{array}{ccccc}
 A^3VW & \xrightarrow{1b^{(2)}} & AVW & & A^3VW & \xrightarrow{1b^{(2)}} & AVW & & A^3VW & \xrightarrow{1b^{(2)}} & AVW \\
 \downarrow 1e_1 11 & & \downarrow 1s & & \downarrow d_1 11 & & \downarrow b^0 & & \downarrow d_1 11 & & \downarrow b^0 \\
 A^3VW & \xrightarrow{1b^{(2)}} & AVW & & A^3VW & \xrightarrow{1b^{(2)}} & AVW & & A^2VW & \xrightarrow{b^{(2)}} & AVW \\
 \downarrow d_1 11 & & \downarrow b^0 & & & & & & \downarrow e_1 11 & & \downarrow s \\
 A^2VW & \xrightarrow{b^{(2)}} & VW & & A^2VW & \xrightarrow{b^{(2)}} & VW & & A^2VW & \xrightarrow{b^{(2)}} & VW
 \end{array}$$

commute by the construction of b^0 and commutativity of the first diagram of Lemma 4.8. Since the left verticals are equal by the fourth and fifth equalities of A.9, and the top rows are equal epimorphisms, we deduce the equality of the right verticals; that is, commutativity of the first diagram of the claim. In the diagram

$$\begin{array}{ccccc}
 A^3VW & \xleftarrow{111b^{(2)}} & A^5VW & \xrightarrow{111b^{(2)}} & A^3VW \\
 & \searrow d_2 1111 & & \searrow 11d_1 11 & \\
 & & A^4VW & & A^4VW \\
 & \swarrow 11b^{(2)} & \searrow m^{(2)} 11 & \swarrow m^{(2)} 11 & \searrow 11b^{(2)} \\
 A^2VW & & A^2VW & & A^2VW \\
 \parallel & & \downarrow b^{(2)} & & \parallel \\
 A^2VW & \xrightarrow{b^{(2)}} & VW & \xleftarrow{b^{(2)}} & A^2VW
 \end{array}$$

the bottom regions commute by the associativity of $b^{(2)}$. The region at the middle commutes by the second equality of A.9. The triangle-shaped region at the top-left commutes by the functoriality of the monoidal product and the triangle-shaped region at the top-right commutes by the construction of b^0 . Since the morphisms of the top row are equal epimorphisms, we deduce the equality of the left-bottom and right-bottom paths. This proves commutativity of the second diagram of the claim. \square

Assume now that idempotent morphisms in \mathcal{C} split. Then in particular e_1 — which is equal to $s: A^2 \rightarrow A^2$ of (4.1) by (4.6) — splits by some epimorphism $\hat{e}_1: A^2 \rightarrow A \circ A$, via some monomorphism $\check{e}_1: A \circ A \rightarrow A^2$; and e_2 splits by some epimorphism $\hat{e}_2: A^2 \rightarrow A \bullet A$,

via some monomorphism $\check{e}_2: A \bullet A \rightarrow A^2$. By the last equality of A.9 and its opposite-coopposite, and by universality of the equalizers in the bottom rows of

$$\begin{array}{ccc} & A^3 & \\ \hat{d}_1 \swarrow & \downarrow d_1 & \searrow \\ A \circ A & \xrightarrow{\check{e}_1} A^2 \xrightarrow[e_1]{1} A^2 & \end{array} \quad \begin{array}{ccc} & A^3 & \\ \hat{d}_2 \swarrow & \downarrow d_2 & \searrow \\ A \bullet A & \xrightarrow{\check{e}_2} A^2 \xrightarrow[e_2]{1} A^2, & \end{array} \quad (4.7)$$

there exist unique morphisms \hat{d}_1 (equal to $\hat{e}_1.d_1$, in fact) and \hat{d}_2 (equal to $\hat{e}_2.d_2$, in fact) rendering commutative the diagrams.

Proposition 4.10. *If we add to the standing assumptions of the section the requirements that idempotent morphisms in \mathcal{C} split and that the morphisms \hat{d}_1 and \hat{d}_2 in (4.7) are regular epimorphisms, then for any modules $v: AV \rightarrow V$ and $w: AW \rightarrow W$, the object $V \circ W$ admits an A -module structure too. Moreover, for any morphisms of A -modules $f: V \rightarrow V'$ and $W \rightarrow W'$, $f \circ g: V \circ W \rightarrow V' \circ W'$ is a morphism of A -modules too.*

Proof. By our assumption on \mathcal{V} being closed under the monoidal product, the object VW belongs to the class \mathcal{V} . Since $\check{s}: V \circ W \rightarrow VW$ is a split monomorphism, it follows by our assumption on \mathcal{V} being closed under retracts that $V \circ W$ also belongs to \mathcal{V} .

Thanks to the commutativity of the first diagram of Lemma 4.9, the left-bottom path coequalizes the parallel morphisms of

$$\begin{array}{ccc} AVW & \xrightarrow[1s]{111} AVW & \xrightarrow{1\hat{s}} A(V \circ W) \\ & \downarrow b^0 & \downarrow b \\ & VW & \xrightarrow{\hat{s}} V \circ W. \end{array}$$

Hence we can use the universality of the coequalizer in the top row to define b . It is an associative A -action since b^0 is: see Lemma 4.9.

Let us see that b is a regular epimorphism. In doing so, we use again the shorthand notation $b^{(2)} := A^2VW \dashv_{1c1} \dashv_{AVW} \dashv_{VW} VW$. In the first commutative diagram of

$$\begin{array}{ccc} A^2VW & \xrightarrow{b^{(2)}} VW & \xrightarrow{\hat{s}} V \circ W \\ \hat{e}_1 11 \downarrow & \searrow e_1 11 & \downarrow s \\ (A \circ A)VW & \xrightarrow{\check{e}_1 11} A^2VW \xrightarrow{b^{(2)}} VW & \xrightarrow{\hat{s}} V \circ W \end{array} \quad \begin{array}{ccc} A^3VW & \xrightarrow{1b^{(2)}} AVW & \xrightarrow{1\hat{s}} A(V \circ W) \\ \downarrow d_1 11 & & \downarrow b^0 \\ \hat{d}_1 11 \downarrow & \searrow e_1 11 & \downarrow b \\ A^2VW & \xrightarrow{b^{(2)}} VW & \xrightarrow{\hat{s}} V \circ W \\ \hat{e}_1 11 \downarrow & & \downarrow s \\ (A \circ A)VW & \xrightarrow{\check{e}_1 11} A^2VW \xrightarrow{b^{(2)}} VW & \xrightarrow{\hat{s}} V \circ W \end{array}$$

(where the middle region commutes by Lemma 4.8), the top row is a composite of regular epimorphisms; hence a regular epimorphism. Since the left column is epi, this shows that the bottom row is a regular epimorphism. Thus also the left-bottom path of the second commutative diagram is a regular epimorphism. Since the top row of the second diagram is an epimorphism, this proves that b is a regular epimorphism.

It remains to prove the non-degeneracy of b on the left with respect to \mathcal{V} . If $b1$ coequalizes the morphisms $1f$ and $1g$ to $A(V \circ W)Y$ for some morphisms f and g to $(V \circ W)Y$ and

$Y \in \mathcal{Y}$, then the equal paths of the diagram

$$\begin{array}{ccccc}
 A^3(V \circ W)Y & \xlongequal{\quad} & A^3(V \circ W)Y & \xrightarrow{11b1} & A^2(V \circ W)Y \\
 \downarrow d_{211} & \searrow 111\check{s}1 & \nearrow 111\check{s}1 & \nearrow 11s1 & \downarrow 11\check{s} \\
 & A^3VWY & \xrightarrow{11b^01} & A^2VWY & \\
 & \parallel & & & \\
 & A^3VWY & \xrightarrow{11b^01} & A^2VWY & \\
 & \downarrow d_2111 & & & \downarrow b^{(2)}1 \\
 A^2(V \circ W)Y & \xrightarrow{11\check{s}1} & A^2VWY & \xrightarrow{b^{(2)}1} & VWY
 \end{array}$$

(where both regions at the bottom-right commute by Lemma 4.9) coequalize the morphisms $111f$ and $111g$ to $A^3(V \circ W)Y$. Then using the factorization (4.7) of d_2 , we conclude that the equal paths around

$$\begin{array}{ccc}
 A^2(V \circ W)Y & \xrightarrow{e_211} & A^2(V \circ W)Y \\
 \downarrow 11\check{s}1 & & \downarrow 11\check{s}1 \\
 11\check{s}1 \left(A^2VWY \right. & \xrightarrow{e_2111} & \left. A^2VWY \right) \\
 \downarrow 11s1 & & \downarrow b^{(2)}1 \\
 A^2VWY & \xrightarrow{b^{(2)}1} & VWY
 \end{array}$$

(where the bottom-right region commutes by Lemma 4.8) coequalize the morphisms $11f$ and $11g$ to $A^2(V \circ W)Y$. Since v and w are non-degenerate on the left with respect to \mathcal{Y} and $\check{s}: V \circ W \rightarrow VW$ is a (split) monomorphism, this proves $f = g$ hence the non-degeneracy of b on the left with respect to \mathcal{Y} .

As for functoriality of the monoidal product \circ , since $A^3VW \xrightarrow{1b^{(2)}} AVW \xrightarrow{1\check{s}} A(V \circ W)$ is an epimorphism, it follows by the commutativity of

$$\begin{array}{ccccc}
 A(V \circ W) & \xrightarrow{1(f \circ g)} & A(V' \circ W') & & \\
 \downarrow b & \swarrow 1\check{s} & \searrow 1\check{s}' & & \downarrow b' \\
 & AVW & \xrightarrow{1fg} & AV'W' & \\
 & \swarrow 1b^{(2)} & \searrow 1b'^{(2)} & & \\
 & A^3VW & \xrightarrow{111fg} & A^3V'W' & \\
 & \downarrow d_111 & & \downarrow d_111 & \\
 & A^2VW & \xrightarrow{11fg} & A^2V'W' & \\
 & \swarrow b^{(2)} & \searrow b'^{(2)} & & \\
 & VW & \xrightarrow{fg} & V'W' & \\
 \downarrow \check{s} & & & & \downarrow \check{s}' \\
 V \circ W & \xrightarrow{f \circ g} & V' \circ W' & &
 \end{array}$$

for any morphisms f and g of A -modules, that $f \circ g$ is a morphism of A -modules. \square

Observe that, in the category of vector spaces, the requirement that \hat{d}_1 and \hat{d}_2 be regular epimorphisms becomes axiom (iv) in [4, Definition 2.1]; equivalently, axiom (vi) of [1, Definition 1.1] that we did not require in Definition 2.10.

Lemma 4.11. *Under the same hypotheses as in Proposition 4.10, the action $b: A(V \circ W) \rightarrow V \circ W$, constructed in the proof of Proposition 4.10, admits the following equivalent characterizations.*

(1) *b is the unique morphism rendering commutative*

$$\begin{array}{ccccccc} A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{11w} & AVW & \xrightarrow{1\hat{s}} & A(V \circ W) \\ t_1 11 \downarrow & & & & & & \downarrow b \\ A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW & \xrightarrow{\hat{s}} & V \circ W. \end{array}$$

(2) *b is the unique morphism rendering commutative*

$$\begin{array}{ccccccc} A^2VW & \xrightarrow{1v1} & AVW & \xrightarrow{1\hat{s}} & A(V \circ W) & & \\ c11 \downarrow & & & & \downarrow b & & \\ A^2VW & & & & & & \\ t_4 11 \downarrow & & & & & & \\ A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{vw} & VW & \xrightarrow{\hat{s}} & V \circ W. \end{array}$$

Proof. The exterior of

$$\begin{array}{ccccccc} A^3VW & \xrightarrow{11c1} & A^2VAW & \xrightarrow{1v11} & AVAW & \xrightarrow{11w} & AVW \xrightarrow{1\hat{s}} A(V \circ W) \\ \downarrow 1c^{-1}11 & & & & \downarrow 1c^{-1}1 & & \downarrow b \\ A^3VW & \xrightarrow{11v1} & A^2VW & & & & \\ \downarrow t_1 111 & & \downarrow t_1 11 & & & & \\ A^3VW & \xrightarrow{11v1} & A^2VW & & & & \\ \downarrow 1c11 & & \downarrow 1c1 & & & & \\ A^3VW & \xrightarrow{11c1} & A^2VAW & \xrightarrow{1v11} & AVAW & & \\ \downarrow m111 & & \downarrow m111 & & \downarrow v11 & & \\ A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{v11} & VAW & \xrightarrow{1w} & VW \xrightarrow{\hat{s}} V \circ W \end{array}$$

commutes by the constructions of b^0 and b . All of the small regions in the left half commute by functoriality of the monoidal product, naturality and coherence of the braiding, and associativity of v . Since $A^3VW \xrightarrow{11c1} A^2VAW \xrightarrow{1v11} AVAW$ is epi, this proves commutativity of the large region on the right; that is, assertion (1).

Similarly, applying (2.6), we obtain the alternative expression

$$A^3 \xrightarrow{c1} A^3 \xrightarrow{t_4 1} A^3 \xrightarrow{1m} A^2$$

of d_1 . Using it, we deduce commutativity of the leftmost region of

$$\begin{array}{ccccccc}
 A^3VW & \xrightarrow{11c1} & A^2VAW & \xrightarrow{111w} & A^2VW & \xrightarrow{1v1} & AVW \xrightarrow{1\hat{s}} A(V \circ W) \\
 \downarrow c111 & & \downarrow c111 & & \downarrow c11 & & \downarrow b \\
 A^3VW & \xrightarrow{11c1} & A^2VAW & & A^2VW & & \\
 \downarrow t_4111 & & \downarrow t_4111 & & \downarrow t_411 & & \\
 A^3VW & \xrightarrow{11c1} & A^2VAW & & A^2VW & & \\
 \downarrow 1m11 & & \downarrow 1c11 & & \downarrow 1c1 & & \\
 & & AVA^2W & \xrightarrow{111w} & AVAW & & \\
 \downarrow 11m1 & & \downarrow 11w & & \downarrow 11w & & \\
 A^2VW & \xrightarrow{1c1} & AVAW & \xrightarrow{11w} & AVW & \xrightarrow{v1} & VW \xrightarrow{\hat{s}} V \circ W.
 \end{array}$$

d_111 (curved arrow from A^3VW to A^2VW)

Since the regions in the left and central columns commute by functoriality of the monoidal product, naturality of the braiding and associativity of w , and since in the top row the composite $A^3VW \xrightarrow{11c1} A^2VAW \xrightarrow{111w} A^2VW$ is epi, this proves mutativity of the large region on the right; that is, part (2). \square

Theorem 4.12. *If we add to the standing assumptions of the section that idempotent morphisms in \mathcal{C} split, and that the morphisms \hat{d}_1 and \hat{d}_2 in (4.7) are regular epimorphisms, then there is a monoidal structure on the category of A -modules for which the functor U in Theorem 4.6 is strict monoidal.*

Proof. In view of Propositions 4.7 and 4.10, we only need to show that the unit and associativity isomorphisms of the monoidal category of L -bicomodules, if evaluated at A -modules, are A -module morphisms.

The object $L \circ V$ — defined up-to isomorphism — can be chosen to be V . With this choice,

$$\check{s} = V \xrightarrow{\tau} LV \quad \text{and} \quad \hat{s} = LV \xrightarrow{1\tau} L^2V \xrightarrow{\mu 1} LV \xrightarrow{\varepsilon 1} V.$$

The left unit constraint is a morphism of A -modules if and only if, for any A -module $v: AV \rightarrow V$, the resulting A action $b: A(L \circ V) = AV \rightarrow V$ is equal to v . By Lemma 4.11 (2) this is equivalent to the commutativity of the large central region of

$$\begin{array}{ccccccc}
 A^2LAV & \xrightarrow{1n_211} & A^3V & \xrightarrow{1\sqcap_1^L 1} & A^2V & & \\
 \downarrow c111 & \searrow 111v & \downarrow 11v & & \downarrow 1v & & \\
 & A^2LV & \xrightarrow{1n_21} & A^2V & \xrightarrow{1p1} & ALV & \xrightarrow{1\hat{s}} AV \\
 & \downarrow c11 & & & & & \downarrow v \\
 & A^2LV & \downarrow t_411 & & & & \\
 & A^2LV & \xrightarrow{b^{(2)}} & LV & \xrightarrow{\hat{s}} & V & \\
 & \parallel & & & & \uparrow v & \\
 & A^2LV & \xrightarrow{1c1} & ALAV & \xrightarrow{n_211} & A^2V & \xrightarrow{\sqcap_1^L 1} AV \\
 & \nearrow 111v & & & & \uparrow 1v & \\
 A^2LAV & \xrightarrow{1c11} & ALA^2V & \xrightarrow{n_2111} & A^3V & \xrightarrow{\sqcap_1^L 11} & A^2V \xrightarrow{m1} AV.
 \end{array} \tag{4.8}$$

Above the central region, the region on the right commutes by the commutativity of

$$\begin{array}{ccccccc}
 A^2V & \xlongequal{\quad} & A^2V & \xleftarrow{1m1} & A^3V & \xrightarrow{1m1} & A^2V \\
 \downarrow 1v & & \downarrow 1\tau 1 & & \downarrow 1t_1 1 & & \downarrow \cap_1^L 1 \\
 & & ALAV & \xleftarrow{1p11} & A^3V & \xrightarrow{\cap_1^L 11} & A^2V & \xrightarrow{j11} & AV \\
 & & \downarrow 11v & & \downarrow 11v & & \downarrow \cap_1^L 1 & & \downarrow v \\
 AV & \xrightarrow{1\tau} & ALV & \xleftarrow{1p1} & A^2V & \xrightarrow{\cap_1^L 1} & AV & \xrightarrow{j1} & V \\
 \downarrow p1 & & \downarrow p11 & & \downarrow p1 & & \downarrow p1 & & \parallel \\
 LV & \xrightarrow{1\tau} & L^2V & \xrightarrow{\mu 1} & LV & \xrightarrow{\varepsilon 1} & V & & V
 \end{array}$$

\hat{s}

(where the top-right region commutes by the second equality of A.8), because the morphisms in the top row are equal epimorphisms. From this we also obtain

$$\begin{array}{ccccccc}
 & & & & b^{(2)} & & \\
 A^2LV & \xrightarrow{1c1} & ALAV & \xrightarrow{n_2 11} & A^2V & \xrightarrow{1v} & AV & \xrightarrow{p1} & LV \\
 & & & & \downarrow \cap_1^L 1 & & & & \downarrow \hat{s} \\
 & & & & AV & \xrightarrow{v} & V & &
 \end{array}$$

proving commutativity of the region below the central region of (4.8). All other regions around the central region commute by functoriality of the monoidal product and the associativity of v . Thus since $111v: A^2LAV \rightarrow A^2LV$ is epi, commutativity of the central region is equivalent to the commutativity of the exterior of (4.8). This holds by the coopposite of the second equality of A.21 (using that $11p11: A^4V \rightarrow A^2LAV$ is epi).

An analogous reasoning traces back the A -module morphism property of the right unit constraint to the first equality of A.21.

The associativity isomorphism is a morphism of A -modules if and only if, for any A -modules $v: AV \rightarrow V$, $w: AW \rightarrow W$ and $z: AZ \rightarrow Z$, the actions $A((V \circ W) \circ Z) \rightarrow (V \circ W) \circ Z$ and $A(V \circ (W \circ Z)) \rightarrow V \circ (W \circ Z)$ coincide (omitting the associativity constraint in the category of L -bicomodules). These actions fit the respective diagrams of Figure 1. The rightmost regions, as well as the bottom regions on their left, commute by parts (1) and (2) of Lemma 4.11. All other regions commute by functoriality of the monoidal product, and naturality and coherence of the braiding. The top rows are equal epimorphisms (up-to the omitted associativity isomorphism of the category of L -bicomodules) and the left-bottom paths are equal by the associativity of the actions v and z , and A.2. Hence the right verticals are equal proving the claim. \square

Note the difference between Theorem 4.12 and [4, Theorem 5.6] in the case when C is the category of vector spaces over a given field: In [4, Theorem 5.6] the weak multiplier bialgebra in question is assumed to be *left full* while in Theorem 4.12 this assumption is replaced by the non-degeneracy of n_1 in (3.6) on the right with respect to the chosen class \mathcal{Y} .

Remark 4.13. In [5], monoidality of the category of modules over a (nice enough) multiplier bimonoid A in a braided monoidal category C was explained by the structure of the induced endofunctor $A(-)$ on C . Namely, it was shown to carry the structure of a *multiplier*

bimonad; a generalization of bimonad (which is another name for opmonoidal monad). Recall that a multiplier bimonad on a monoidal category is an endofunctor T equipped with a morphism $T_0: T(I) \rightarrow I$ and natural transformations

$$T(XT(Y)) \xrightarrow{\overleftarrow{T}_2} T(X)T(Y) \xleftarrow{\overrightarrow{T}_2} T(T(X)Y)$$

subject to compatibility conditions in [5].

A similar explanation of the monoidality of the category of modules over a (nice enough) *weak* multiplier bimonoid A is possible, in fact, but the treatment is technically more involved. For this reason, we sketch here the construction without a detailed proof; leaving the technicalities to the interested reader.

In the setting of Theorem 3.4, the base object L of a regular weak multiplier bimonoid A carries the structure $(\delta, \varepsilon, \mu)$ of a coseparable comonoid; so in particular that of a semigroup $\mu: L^2 \rightarrow L$. Using the braiding in the base category, any L -bicomodule can equivalently be regarded as a left comodule over the monoidal product comonoid LL^{op} , where L^{op} is the comonoid with the same underlying object L , the same counit ε but the opposite comultiplication $c^{-1}.\delta$. The comonoid LL^{op} inherits a coseparable structure of L . Hence as explained in Remark 3.6, any left LL^{op} -comodule can equivalently be regarded as a firm left module over the monoidal product semigroup LL^{op} , where the semigroup L^{op} has the same underlying object L and opposite multiplication $\mu.c$. This yields an isomorphism between the category of L -bicomodules and the category of firm left LL^{op} -modules. Since the category of L -bicomodules is monoidal — via the monoidal product \circ and the monoidal unit L — this isomorphism induces a monoidal structure — also to be denoted by (\circ, L) — on the category of firm left LL^{op} -modules.

For a regular weak multiplier bimonoid A , in addition to the L -actions n_1 and n_2 in (3.6), we can introduce two more actions

$$\begin{array}{ccc} A^3 & \xrightleftharpoons[\sqcap_1^R \cdot 1.c^{-1}1]{\sqcap_1^L 1} & A^2 \xrightarrow{p1} LA \\ & & \downarrow \bar{n}_1 \\ & & A \end{array} \quad \begin{array}{ccc} A^3 & \xrightleftharpoons[1\sqcap_1^R \cdot 1c^{-1}]{1\sqcap_1^L} & A^2 \xrightarrow{1p} AL \\ & & \downarrow \bar{n}_2 \\ & & A \end{array}$$

of the opposite semigroup L^{op} . All four of these actions are associative, and all commute with each other. Hence they make A an LL^{op} -bimodule, with the left and right actions

$$L^2A \xrightarrow{1\bar{n}_1} LA \xrightarrow{n_1} A \quad AL^2 \xrightarrow{n_21} AL \xrightarrow{\bar{n}_2} A.$$

There is an important difference between the left and right actions. In the setting of Proposition 4.4, n_1 and \bar{n}_1 are firm actions: they correspond (in the way described in Remark 3.7) to the coactions τ and $c^{-1}.\bar{\tau}$, respectively. On the contrary, n_2 and \bar{n}_2 need not be so without further assumptions. In other words, the LL^{op} -bimodule A is firm on the left but not necessarily on the right.

Take now any left LL^{op} -module X (with action $x: L^2X \rightarrow X$). We can define an object $A \boxtimes X$ as the usual LL^{op} -module tensor product of the right LL^{op} -module A and the left LL^{op} -module X ; that is, as the coequalizer

$$AL^2X \xrightleftharpoons[1x]{\bar{n}_21.n_211} AX \twoheadrightarrow A \boxtimes X.$$

The firm left LL^{op} -action $n_1.1\bar{n}_1$ on A induces a firm left LL^{op} -action on $A \boxtimes X$ (regardless the properties of the left LL^{op} -module X). In particular, there is an endofunctor $A \boxtimes (-)$ on the category of firm left LL^{op} -modules.

Now the endofunctor $A \boxtimes (-)$ on the monoidal category of firm left LL^{op} -modules can be equipped with the structure of a multiplier bimonad. For any left LL^{op} -modules X and Y , the structure morphisms of this multiplier bimonad are constructed using the universality of the coequalizers in the top rows of

$$\begin{array}{ccccc}
 AXAY \twoheadrightarrow A \boxtimes (X \circ (A \boxtimes Y)) & A^2XY \twoheadrightarrow A \boxtimes ((A \boxtimes X) \circ Y) & AL \twoheadrightarrow A \boxtimes L \\
 1c^{-1}1 \downarrow & c11 \downarrow & \downarrow n_2 \\
 A^2XY & A^2XY & \downarrow \\
 t_111 \downarrow & t_411 \downarrow & \downarrow \\
 A^2XY & A^2XY & \downarrow \\
 1c1 \downarrow & 1c1 \downarrow & \downarrow \\
 AXAY \twoheadrightarrow (A \boxtimes X) \circ (A \boxtimes Y) & AXAY \twoheadrightarrow (A \boxtimes X) \circ (A \boxtimes Y) & A \twoheadrightarrow_p L.
 \end{array}$$

5. UNIQUENESS OF THE IDEMPOTENT MORPHISMS AND THE COUNT OF A REGULAR WEAK MULTIPLIER BIMONOID

Consider regular weak multiplier bimonoids $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ and $(t'_1, t'_2, t'_3, t'_4, e'_1, e'_2, j')$ on the same underlying object A . By Corollary 2.5, we know that if $t_1 = t'_1$ and the multiplications $j1.t_1$ and $j'1.t'_1$ are equal and non-degenerate with respect to some class containing I , A , and A^2 , then $t_2 = t'_2$, $t_3 = t'_3$, and $t_4 = t'_4$. The aim of this section is to find criteria for the uniqueness of the remaining structure e_1, e_2, j . The findings below generalize [25, Lemma 3.3] and [4, Theorem 2.8].

Lemma 5.1. *Let \mathcal{C} be a braided monoidal category in which coequalizers are preserved by the monoidal product. Consider a regular weak multiplier bimonoid $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ in \mathcal{C} such that the induced multiplication $m := j1.t_1$ is a regular epimorphism and non-degenerate with respect to some class of objects containing I , A and A^2 . Then the following hold.*

- (1) *There is a unique morphism $g: A^2 \rightarrow A^2$ rendering commutative the equivalent diagrams*

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1m} & A^2 \\
 1t_1 \downarrow & & \downarrow g \\
 A^3 & \xrightarrow{\overline{\Pi}_2^R 1} & A^2
 \end{array}
 \quad
 \begin{array}{ccc}
 A^3 & \xrightarrow{m1} & A^2 \\
 t_21 \downarrow & & \downarrow g \\
 A^3 & \xrightarrow{1\overline{\Pi}_1^L} & A^2.
 \end{array}$$

- (2) *The following diagrams commute.*

$$\begin{array}{ccccc}
 A^3 & \xrightarrow{g1} & A^3 & \xrightarrow{1c} & A^3 \\
 1c \downarrow & & & & \downarrow m1 \\
 A^3 & \xrightarrow{1e_1} & A^3 & \xrightarrow{m1} & A^2
 \end{array}
 \quad
 \begin{array}{ccccc}
 A^3 & \xrightarrow{1g} & A^3 & \xrightarrow{c1} & A^3 \\
 c1 \downarrow & & & & \downarrow 1m \\
 A^3 & \xrightarrow{e_21} & A^3 & \xrightarrow{1m} & A^2
 \end{array}$$

(3) The following diagrams commute.

$$\begin{array}{ccc}
 A^3 & \xrightarrow{1g} & A^3 \\
 m1 \downarrow & & \downarrow m1 \\
 A^2 & \xrightarrow{g} & A^2
 \end{array}
 \quad
 \begin{array}{ccc}
 A^3 & \xrightarrow{g1} & A^3 \\
 1m \downarrow & & \downarrow 1m \\
 A^2 & \xrightarrow{g} & A^2
 \end{array}$$

(4) The following diagrams commute.

$$\begin{array}{ccc}
 A^2 & \xrightarrow{t_1} & A^2 \\
 g \downarrow & \searrow \pi_2^R & \downarrow 1j \\
 A^2 & \xrightarrow{1j} & A
 \end{array}
 \quad
 \begin{array}{ccc}
 A^2 & \xrightarrow{t_2} & A^2 \\
 g \downarrow & \searrow \pi_1^L & \downarrow j1 \\
 A^2 & \xrightarrow{j1} & A
 \end{array}$$

Proof. (1) It follows by A.4 and the non-degeneracy of m with respect to A that the left-bottom path of the first diagram coequalizes any morphisms that the top row does. Since the top row is a regular epimorphism, there is a unique morphism g as in the first diagram. It obeys

where the first equality holds by definition of g . Since $11m: A^4 \rightarrow A^3$ is epi, this proves commutativity of the second diagram (which is the opposite-coopposite of the first one).

(2) The first diagram commutes by

and since $1m1: A^4 \rightarrow A^3$ is epi, where the unlabelled equality holds by definition of g . The second diagram commutes by the opposite-coopposite reasoning.

(3) The left-bottom path of the first diagram in part (1) is a left A -module morphism by (2.2). Hence so is the top-right path. Since $11m: A^4 \rightarrow A^3$ is also left A -linear and epi, this proves commutativity of the first diagram. The second diagram commutes by the opposite-coopposite reasoning.

(4) The first diagram commutes by

and the non-degeneracy of m . The first equality holds by Axiom VII and the second by commutativity of the first diagram in part (2). The second diagram commutes by the opposite-coopposite reasoning. \square

Theorem 5.2. *Let \mathcal{C} be a braided monoidal category in which coequalizers are preserved by the monoidal product. Let $(t_1, t_2, t_3, t_4, e_1, e_2, j)$ and $(t_1, t_2, t_3, t_4, e'_1, e'_2, j')$ be regular weak multiplier bimonoids in \mathcal{C} such that the induced multiplications $m := j1.t_1$ and $m' := j'1.t_1$ are equal regular epimorphisms and non-degenerate with respect to some class of objects containing I , A and A^2 . Then the following are equivalent.*

- (i) $e_1 = e'_1$,
- (ii) $e_2 = e'_2$,
- (iii) $g = g'$,
- (iv) $j = j'$.

These equivalent assertions hold true if in addition idempotent morphisms in \mathcal{C} split and the morphisms \hat{d}_1 (or \hat{d}_2) of (4.7) are epimorphisms both for the unprimed and for the primed data.

Proof. (i) \Leftrightarrow (ii). This follows by (2.18), applied both to the unprimed and the primed data, and the non-degeneracy of $m = m'$ with respect to A .

(i) \Leftrightarrow (iii). This follows by the first diagram in part (2) of Lemma 5.1, applied both to the unprimed and the primed data, and the non-degeneracy of $m = m'$ with respect to A .

(iv) \Rightarrow (iii). If (iv) holds then the left-bottom path of the first diagram of Lemma 5.1 (1) is the same for the primed and for the unprimed data. Then so is the top-right path. Since $1m$ and $1m'$ are equal epimorphisms, this proves $g = g'$.

(iii) \Rightarrow (iv). This follows by the fact that m and m' are equal epimorphisms and the calculation

$$j.m' \stackrel{(2.14)}{=} j'j.t_1 = j'j.g \stackrel{(iii)}{=} j'j.g' = j'j.t_2 \stackrel{(2.15)}{=} j'.m,$$

in which the unlabelled equalities hold by Lemma 5.1(4).

Assume now that idempotent morphisms in \mathcal{C} split. The morphisms d_1 of A.9, built up from the unprimed and from the primed data, are the same. Applying the last equality of A.9 to the unprimed and the primed data, respectively, we get

$$e_1.d_1 = d_1 \qquad e'_1.d_1 = d_1. \tag{5.1}$$

Using the second equality, we obtain the equivalent expressions of d_1 in

$$d_1 = e'_1.d_1 = \check{e}'_1.\check{e}'_1.d_1 = \check{e}'_1.\hat{d}'_1.$$

Substituting this expression of d_1 in the first equality of (5.1), we get $e_1.\check{e}'_1.\hat{d}'_1 = \check{e}'_1.\hat{d}'_1$. Since \hat{d}'_1 is epi by assumption, this shows that \check{e}'_1 equalizes e_1 and 1 . Thus universality of

the equalizer

$$\begin{array}{ccc}
 & A \circ' A & \\
 f \swarrow & \downarrow \check{e}'_1 & \\
 A \circ A & \xrightarrow{\check{e}_1} A^2 & \xrightarrow[e_1]{1} A^2
 \end{array} \tag{5.2}$$

yields a morphism f as in the diagram. It is an isomorphism with the inverse constructed by a symmetrical reasoning.

Post-composing the morphisms around the triangle of (5.2) by \check{e}'_1 gives $\check{e}'_1 \cdot \check{e}_1 \cdot f = \check{e}'_1 \cdot \check{e}'_1 = 1$; so that $\check{e}'_1 \cdot \check{e}_1$ is the inverse of f and therefore $f \cdot \check{e}'_1 \cdot \check{e}_1 = 1$. Using this in the penultimate equality and commutativity of the triangular region of (5.2) in the second one, we obtain

$$e'_1 \cdot e_1 = \check{e}'_1 \cdot \check{e}_1 \cdot \check{e}_1 \cdot \hat{e}_1 = \check{e}_1 \cdot f \cdot \check{e}'_1 \cdot \check{e}_1 \cdot \hat{e}_1 = \check{e}_1 \cdot \hat{e}_1 = e_1.$$

Since e_1 on the right hand side is the first component of an \mathbb{M} -morphism $A^2 \rightarrowtail A^2$ by (2.18), so must be $e'_1 \cdot e_1$ on the left hand side. The \mathbb{M} -morphism with first component $e_1 = e'_1 \cdot e_1$ has $e_2 = e_2 \cdot e'_2$ as the second component, again by (2.18), applied both to the primed and the unprimed data.

Symmetrical reasoning leads to the further equalities

$$e_1 \cdot e'_1 = e'_1 \text{ and } e'_2 \cdot e_2 = e'_2, \quad e'_2 \cdot e_2 = e_2 \text{ and } e_1 \cdot e'_1 = e_1, \quad e_2 \cdot e'_2 = e'_2 \text{ and } e'_1 \cdot e_1 = e'_1.$$

They immediately imply $e_1 = e_1 \cdot e'_1 = e'_1$; that is, the first of the equivalent assertions of the theorem. \square

6. WORKING IN A CLOSED BRAIDED MONOIDAL CATEGORY

In this final section we assume that the braided monoidal category \mathcal{C} is also closed and investigate the consequences of this on the assumptions and the constructions of the previous sections.

A braided monoidal category \mathcal{C} is said to be *closed* if, for any object X , the functor $X(-): \mathcal{C} \rightarrow \mathcal{C}$ possesses a right adjoint, to be denoted by $[X, -]$ (this is equivalent to $(-)X \cong X(-)$ possessing a right adjoint). The unit and the counit of the adjunction $X(-) \dashv [X, -]$ will be denoted by coev and ev , respectively.

Since any left adjoint functor preserves coequalizers, in a closed braided monoidal category coequalizers are preserved by taking the monoidal product with any object.

6.1. The multiplier monoid. In this section we recall some background material from [6]. Consider the images of the morphisms

$$A^2[A, A] \xrightarrow{1\text{ev}} A^2 \xrightarrow{m} A \quad A^2[A, A] \xrightarrow{1c} A[A, A]A \xrightarrow{\text{ev}1} A^2 \xrightarrow{m} A$$

under the adjunction isomorphism $\mathcal{C}(A^2[A, A], A) \cong \mathcal{C}([A, A], [A^2, A])$. If their pullback exists, as it will in any abelian category, then we call it the *multiplier monoid* of A (a term justified by the verification of its monoid structure in [6]) and we denote it by $\mathbb{M}(A)$ as in the first diagram of

$$\begin{array}{ccc}
 \mathbb{M}(A) & \dashv \multimap & [A, A] \\
 \downarrow & & \downarrow \\
 [A, A] & \longrightarrow & [A^2, A]
 \end{array}
 \quad
 \begin{array}{ccc}
 A\mathbb{M}(A)A & \dashv \multimap & A^2 \\
 h_2 1 \downarrow & & \downarrow m \\
 A^2 & \xrightarrow{m} & A.
 \end{array} \tag{6.1}$$

Using the adjunction isomorphism $\mathcal{C}(\mathbb{M}(A), [A^2, A]) \cong \mathcal{C}(A^2\mathbb{M}(A), A)$, the first diagram in (6.1) translates to the second one, involving the morphisms $(\mathbb{M}(A)A \xrightarrow{h_1} A \xleftarrow{h_2} A\mathbb{M}(A))$ with the universal property that for any object X , the morphisms $f: X \rightarrow \mathbb{M}(A)$ correspond bijectively to pairs of morphisms $(XA \xrightarrow{f_1} A \xleftarrow{f_2} AX)$ such that

$$\begin{array}{ccc} XA & \xrightarrow{f_1} & A \\ f_2 \downarrow & & \downarrow f \\ \mathbb{M}(A)A & \xrightarrow{h_1} & A \end{array} \quad (6.2)$$

commutes; that is f_1 and f_2 are components of an \mathbb{M} -morphism $X \rightarrow A$. We call f_1 and f_2 the *components* also of the corresponding morphism $f: X \rightarrow \mathbb{M}(A)$ in \mathcal{C} . Explicitly, the correspondence between $f: X \rightarrow \mathbb{M}(A)$ and its components (f_1, f_2) is expressed by the commutative diagrams

$$\begin{array}{ccccc} XA & \xrightarrow{f_1} & A & \xleftarrow{f_2} & AX \\ f_1 \downarrow & & \parallel & & \downarrow f \\ \mathbb{M}(A)A & \xrightarrow{h_1} & A & \xleftarrow{h_2} & A\mathbb{M}(A). \end{array} \quad (6.3)$$

The pair $(\mathbb{M}(A)A \xrightarrow{h_1} A \xleftarrow{h_2} A\mathbb{M}(A))$ in (6.1) can be regarded as the components of the identity morphism $\mathbb{M}(A) \rightarrow \mathbb{M}(A)$. By the associativity of m , $(A^2 \xrightarrow{m} A \xleftarrow{m} A^2)$ are components of a morphism $i: A \rightarrow \mathbb{M}(A)$.

Proposition 6.1. *If the pullback $\mathbb{M}(A)$ exists for a semigroup A with non-degenerate multiplication m , then h_1 in (6.3) is non-degenerate on the right and h_2 in (6.3) is non-degenerate on the left.*

Proof. For morphisms $f, g: X \rightarrow \mathbb{M}(A)$, it follows by (6.2) that $f_1 = h_1 \cdot f_1$ and $g_1 = h_1 \cdot g_1$ are equal if and only if $f_2 = h_2 \cdot f_2$ and $g_2 = h_2 \cdot g_2$ are equal. If this is the case, then $f = g$ by the universality of the pullback. \square

In the category of vector spaces, the non-degeneracy properties of h_1 and h_2 in Proposition 6.1 are referred to as the *density* of A in $\mathbb{M}(A)$, see [13].

Corollary 6.2. *For a semigroup A satisfying the conditions in Proposition 6.1, and some morphism $f: X \rightarrow \mathbb{M}(A)$, the following are equivalent.*

- f is a monomorphism,
- f_1 is non-degenerate on the right,
- f_2 is non-degenerate on the left.

In particular, for such a semigroup A , the morphism $i: A \rightarrow \mathbb{M}(A)$ — whose components are equal to m — is a monomorphism.

6.2. A distinguished class of objects. In a closed braided monoidal category \mathcal{C} , we can make the following choice of a class \mathcal{Y} of objects in \mathcal{C} . Let \mathcal{Y} contain those objects Y in \mathcal{C} which obey the following properties.

- (a) The functor $Y(-): \mathcal{C} \rightarrow \mathcal{C}$ preserves monomorphisms.

- (b) For any objects X and Z of \mathcal{C} , $q := [X, Z]Y \xrightarrow{\text{coev}} [X, X[X, Z]Y] \xrightarrow{[X, \text{ev}1]} [X, ZY]$ is a monomorphism.

Example 6.3. In the closed symmetric monoidal category of modules over a commutative ring, property (a) characterizes the *flat* modules Y . Property (b) holds for *locally projective* [27] modules Y . Since locally projective modules are also flat, all locally projective (so in particular all projective) modules belong to the class \mathcal{Y} .

From this immediately follows the following.

Example 6.4. In the closed symmetric monoidal category of vector spaces, every object belongs to the class \mathcal{Y} .

More generally, we shall see that in the closed symmetric monoidal category of group graded vector spaces every object belongs to the class \mathcal{Y} . We do this in the following, still more general, setting. Let \mathcal{C} be a closed braided monoidal category and let G be a co-commutative bimonoid in it. Then the category \mathcal{C}^G of G -comodules (that is, the Eilenberg-Moore category of the comonad $G(-)$ on \mathcal{C}) is a braided monoidal category (via the braided monoidal structure lifted from \mathcal{C}). Consequently, in this case any G -comodule $Z \xrightarrow{-z} GZ$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^G & \xrightarrow{Z(-)} & \mathcal{C}^G \\ U \downarrow & & \downarrow U \\ \mathcal{C} & \xrightarrow{Z(-)} & \mathcal{C} \end{array} \quad (6.4)$$

in which U denotes the forgetful functor and in which U and $Z(-)$ in the bottom row are left adjoint functors. Then it follows by a dual version of the *Adjoint Lifting Theorem* [16, Theorem 2] that there is an adjunction $Z(-) \dashv \llbracket Z, - \rrbracket : \mathcal{C}^G \rightarrow \mathcal{C}^G$ whenever the equalizer

$$\llbracket Z, Y \rrbracket \xrightarrow{g} G[Z, Y] \xrightarrow[1\tilde{z}]{1[Z, y]} G[Z, GY]$$

in \mathcal{C} exists, for any G -comodules $Z \xrightarrow{-z} GZ$ and $Y \xrightarrow{-y} GY$, where \tilde{z} is the mate of z under the adjunction $Z(-) \dashv \llbracket Z, - \rrbracket : \mathcal{C} \rightarrow \mathcal{C}$. In particular, \mathcal{C}^G is closed whenever equalizers of coreflexive pairs exist in \mathcal{C} .

Proposition 6.5. *Consider a closed braided monoidal category \mathcal{C} in which the equalizers of coreflexive pairs exist. Let G be a cocommutative Hopf monoid in \mathcal{C} such that the functor $G(-) : \mathcal{C} \rightarrow \mathcal{C}$ preserves monomorphisms. Then an object $Z \xrightarrow{-z} GZ$ of \mathcal{C}^G belongs to the class \mathcal{Y} in the closed braided monoidal category \mathcal{C}^G whenever Z belongs to \mathcal{Y} in \mathcal{C} .*

Proof. Property (a). By the assumption that $G(-) : \mathcal{C} \rightarrow \mathcal{C}$ preserves monomorphisms, so does the forgetful functor $U : \mathcal{C}^G \rightarrow \mathcal{C}$ and therefore the equal paths around the diagram of (6.4). Since U is faithful it also reflects monomorphisms proving that the functor in the top row of (6.4) preserves monomorphisms.

Property (b). Denote by $\delta : G \rightarrow G^2$, $\varepsilon : G \rightarrow I$ and $\mu : G^2 \rightarrow G$ the comultiplication, the counit and the multiplication of the Hopf monoid G , respectively, and for any G -comodules $X \xrightarrow{-x} GX$, $Y \xrightarrow{-y} GY$ and $Z \xrightarrow{-z} GZ$, denote by $a : \llbracket X, Y \rrbracket Z \rightarrow G\llbracket X, Y \rrbracket Z$ the (diagonal)

G -coaction. The left vertical of the commutative diagram

$$\begin{array}{ccccccc}
 G[X, Y]Z & \xlongequal{\quad} & G[X, Y]Z & \xleftarrow{g^1} & \llbracket X, Y \rrbracket Z & \xrightarrow{\text{coev}} & \llbracket X, X \llbracket X, Y \rrbracket Z \rrbracket \xrightarrow{\llbracket X, \text{ev} \rrbracket 1} \llbracket X, YZ \rrbracket \\
 \downarrow 11z & & \downarrow \delta 1z & & \downarrow a & & \downarrow g \\
 G[X, Y]GZ & \xleftarrow{1\varepsilon 11} & G^2[X, Y]GZ & & G\llbracket X, Y \rrbracket Z & \xrightarrow{1\text{coev}} & G[X, X \llbracket X, Y \rrbracket Z] \\
 \downarrow 1c1 & & \downarrow 1c_{G[X, Y], G^1} & & \downarrow 1g1 & & \downarrow 1[X, 1g1] \\
 G^2[X, Y]Z & & G^3[Z, Y]Z & \xrightarrow{\mu^{111}} & G^2[X, Y]Z & & G[X, XG[X, Y]Z] \\
 \downarrow \mu 11 & & & & \downarrow 1\varepsilon 11 & & \downarrow 1[X, 1\varepsilon 11] \\
 G[X, Y]Z & \xlongequal{\quad} & G[X, Y]Z & \xrightarrow{1\text{coev}} & G[X, X \llbracket X, Y \rrbracket Z] & \xrightarrow{1[X, \text{ev} \rrbracket 1]} & G[X, YZ]
 \end{array}$$

is an isomorphism since G is a Hopf monoid. Since the object Z of \mathcal{C} belongs to \mathcal{Y} , $q: [X, Y]Z \rightarrow [X, YZ]$ is a monomorphism. The bottom row is its image under the functor $G(-)$ hence it is a monomorphism. The left pointing arrow of the top row is the image of the (regular) monomorphism g under the functor $G(-)$ hence it is a monomorphism. Thus the left path around the diagram is a monomorphism, proving that also the right pointing path of the top row is a monomorphism. \square

Since the category of vector spaces graded by a group G is isomorphic to the abelian category of comodules over the cocommutative Hopf algebra spanned by G , from Proposition 6.5 and Example 6.4 we obtain the following.

Example 6.6. In the closed symmetric monoidal category of group graded vector spaces every object belongs to the class \mathcal{Y} .

Example 6.7. In any closed braided monoidal category, an object which possesses a (left, equivalently, right) dual, belongs to the class \mathcal{Y} . Indeed, if V^* is the dual of some object V , then the functor $V^*(-) \cong [V, -]$ is right adjoint, hence it preserves monomorphisms proving property (a). The canonical morphism q in part (b) is equal to the composite

$$[X, Y]V^* \xrightarrow{c} V^*[X, Y] \xrightarrow{\cong} [V, [X, Y]] \xrightarrow{\cong} [XV, Y] \xrightarrow{[c^{-1}, 1]} [VX, Y] \xrightarrow{\cong} [X, [V, Y]] \xrightarrow{\cong} [X, V^*Y] \xrightarrow{[X, c^{-1}]} [X, YV^*]$$

hence it is an isomorphism, for any objects X, Y .

Example 6.8. In the closed symmetric monoidal category of complete bornological vector spaces [15, 18], any object Y obeying the *approximation property* belongs to the class \mathcal{Y} . Indeed, property (a) is asserted in Lemma 2.2 of [26], whose Lemma 2.3 discusses a particular case of property (b) (when Z is the base field). A similar reasoning yields property (b) also for any complete bornological vector spaces X and Z ; and Y with the approximation property.

Lemma 6.9. *The full subcategory of \mathcal{C} with objects in \mathcal{Y} , is a monoidal subcategory. That is, the following assertions hold.*

- (1) *The monoidal unit belongs to \mathcal{Y} .*
- (2) *If both objects Y and Y' belong to \mathcal{Y} then so does their monoidal product YY' .*

Proof. (1) follows since $I(-)$ is naturally isomorphic to the identity functor and for $Y = I$ the morphism q is an isomorphism built from the right unit constraints.

(2) Property (a) holds since $YY'(-)$ is naturally isomorphic to the composite of the functors $Y'(-)$ and $Y(-)$. In order to see that property (b) holds, note first commutativity of the diagram

$$\begin{array}{ccc}
 ZY & \xrightarrow{\text{coev}} & [X, XZY] \\
 \text{coev1} \downarrow & & \downarrow [X, 1\text{coev1}] \\
 [X, XZ]Y & \xrightarrow{\text{coev}} & [X, X[X, XZ]Y] \xrightarrow{[X, \text{ev1}]} [X, XZY] \\
 & \searrow q & \nearrow
 \end{array} \quad (6.5)$$

for any objects X, Y, Z . Using this together with the naturality of q we deduce the commutativity of

$$\begin{array}{ccccc}
 & & q^1 & & \\
 [X, Z]YY' & \xrightarrow{\text{coev1}} & [X, X[X, Z]Y]Y' & \xrightarrow{[X, \text{ev1}]1} & [X, ZY]Y' \\
 \parallel & & q \downarrow & & \downarrow q \\
 [X, Z]YY' & \xrightarrow{\text{coev}} & [X, X[X, Z]YY'] & \xrightarrow{[X, \text{ev1}]1} & [X, ZYY'] \\
 & \searrow q & & \nearrow &
 \end{array}$$

The top row is a monomorphism since q is so and $(-)^{Y'} \cong Y'(-)$ preserves monomorphisms. Since the right vertical is also mono, this proves that the bottom row is so. \square

A morphism $W \multimap_j V$ in a braided monoidal category is said to be a *pure monomorphism* if $XW \multimap_{1j} XV$ is a monomorphism for any object X ; of course we see on taking $X = I$ that j is a monomorphism. In particular, any split monomorphism is pure.

Lemma 6.10. *Let $W \multimap_j V$ be a pure monomorphism in \mathcal{C} . If V belongs to \mathcal{Y} then so does W .*

Proof. In order to check property (a) of W , consider a monomorphism $X \multimap_f Y$. Then in the commutative diagram

$$\begin{array}{ccc}
 WX & \xrightarrow{1f} & WY \\
 j1 \downarrow & & \downarrow j1 \\
 VX & \xrightarrow{1f} & VY
 \end{array}$$

the left-bottom path is a monomorphism by our assumptions. Then so is the top-right path and therefore the top row.

As for property (b) of W , for any objects X and Z the left-bottom path in the commutative diagram

$$\begin{array}{ccc}
 [X, Z]W & \xrightarrow{q} & [X, ZW] \\
 1j \downarrow & & \downarrow [X, Zj] \\
 [X, Z]V & \xrightarrow{q} & [X, ZV]
 \end{array}$$

is a monomorphism. Then so is the top-right path and thus the top row. \square

Lemma 6.10 tells us that, in particular, the class \mathcal{Y} is closed under retracts.

Proposition 6.11. *For any morphism $v: ZV \rightarrow W$ in \mathcal{C} , the following assertions are equivalent.*

(i) *For any object X , the map*

$$C(X, V) \rightarrow C(ZX, W), \quad f \mapsto ZX \xrightarrow{1f} ZV \xrightarrow{v} W$$

is injective; that is, v is non-degenerate on the left.

(ii) $V \xrightarrow{\text{coev}} [Z, ZV] \xrightarrow{[Z, v]} [Z, W]$ *is a monomorphism.*

(iii) *For any object X , and any object Y in \mathcal{Y} , the map*

$$C(X, VY) \rightarrow C(ZX, WY), \quad g \mapsto ZX \xrightarrow{1g} ZVY \xrightarrow{v1} WY$$

is injective; that is, v is non-degenerate on the left with respect to \mathcal{Y} .

(iv) *For any object Y in \mathcal{Y} , $VY \xrightarrow{\text{coev}} [Z, ZVY] \xrightarrow{[Z, v1]} [Z, WY]$ is a monomorphism.*

Proof. Composing the map of part (iii) with the isomorphism $C(ZX, WY) \cong C(X, [Z, WY])$, we obtain $C(X, [Z, v1].\text{coev})$. This proves (iii) \Leftrightarrow (iv). Applying it to $Y = I$ proves (i) \Leftrightarrow (ii).

(iv) \Rightarrow (ii) follows by Lemma 6.9 (1) putting $Y = I$.

(ii) \Rightarrow (iv). The diagram

$$\begin{array}{ccccc} VY & \xrightarrow{\text{coev}1} & [Z, ZV]Y & \xrightarrow{[Z, v]1} & [Z, W]Y \\ \parallel & & \downarrow q & & \downarrow q \\ VY & \xrightarrow{\text{coev}} & [Z, ZVY] & \xrightarrow{[Z, v1]} & [Z, WY] \end{array}$$

commutes by the naturality of q and (6.5). The top row is mono by (ii) and the fact that $(-)Y \cong Y(-)$ preserves monos. Since the right vertical is also mono, this proves that the bottom row is so. \square

6.3. The base object of a regular weak multiplier bimonoid. If the semigroup A underlying a regular weak multiplier bimonoid in a closed braided monoidal category \mathcal{C} admits a multiplier monoid $\mathbb{M}(A)$, then the \mathbb{M} -morphisms with components in (3.1), (3.2), (3.3) and (3.4) determine morphisms $\overline{\square}^R, \overline{\square}^L, \square^L$ and \square^R , respectively, all of them from A to $\mathbb{M}(A)$.

Furthermore, the components n_1 and n_2 in (3.6) of an \mathbb{M} -morphism $L \dashv A$ determine a morphism $n: L \rightarrow \mathbb{M}(A)$. It obeys

$$h_1.n1.p1 = n_1.p1 = \square_1^L = h_1.\square^L 1.$$

Hence by the non-degeneracy of h_1 on the right (see Proposition 6.1), \square^L factorizes through the regular epimorphism p via the morphism n . This morphism n is monic if and only if n_1 is non-degenerate on the right (see Corollary 6.2).

Consider a regular weak multiplier bialgebra over a field (that is, a regular weak multiplier bimonoid A in the closed symmetric monoidal category of vector spaces such that the morphisms \hat{d}_1 and \hat{d}_2 in (4.7) are surjective; i.e. regular epimorphisms). Under the additional assumption that A is left full, we saw in Remark 3.7 that $p: A \rightarrow L$ differs by an isomorphism from the corestriction of \square^L to $A \rightarrow \text{Im}(\square^L)$. Since $n: L \rightarrow \mathbb{M}(A)$ differs by the same isomorphism from the canonical inclusion $\text{Im}(\square^L) \rightarrow \mathbb{M}(A)$, we conclude that in this case n is a monomorphism, equivalently, n_1 is non-degenerate on the right (with respect to any vector space, see Proposition 6.11 and Example 6.4).

6.4. Non-degeneracy in the category of Hilbert spaces. Although the category Hilb of complex Hilbert spaces and continuous maps in Example 3.3 is not closed, the findings of this section can be used to describe non-degenerate morphisms therein.

Proposition 6.12. *Let $v: Z \hat{\otimes} V \rightarrow W$ be a morphism in Hilb , and write $i: Z \otimes V \rightarrow Z \hat{\otimes} V$ for the canonical inclusion. The following conditions are equivalent:*

- (i) $Z \hat{\otimes} V \xrightarrow{v} W$ is non-degenerate in Hilb .
- (ii) $Z \otimes V \xrightarrow{i} Z \hat{\otimes} V \xrightarrow{v} W$ is non-degenerate in Vect .
- (iii) $Z \otimes V \xrightarrow{i} Z \hat{\otimes} V \xrightarrow{v} W$ is non-degenerate in Vect with respect to the class of all complex vector spaces.
- (iv) $Z \hat{\otimes} V \xrightarrow{v} W$ is non-degenerate in Hilb with respect to the class of all complex Hilbert spaces.

Proof. (i) \Rightarrow (ii). By hypothesis, the map $\text{Hilb}(X, V) \rightarrow \text{Hilb}(Z \hat{\otimes} X, W)$,

$$f \mapsto Z \hat{\otimes} X \xrightarrow{1 \hat{\otimes} f} Z \hat{\otimes} V \xrightarrow{v} W \quad (6.6)$$

is injective. We should prove that the map $\text{Vect}(X, V) \rightarrow \text{Vect}(Z \otimes X, W)$,

$$f \mapsto Z \otimes X \xrightarrow{1 \otimes f} Z \otimes V \xrightarrow{i} Z \hat{\otimes} V \xrightarrow{v} W \quad (6.7)$$

is injective for every complex vector space X . But, due to the fact that vector spaces are all direct sums of copies of the base field, and that the algebraic tensor product preserves direct sums, we see that it is enough to check the injectivity for $X = \mathbb{C}$. Now, \mathbb{C} is certainly a Hilbert space, and every linear map $f: \mathbb{C} \rightarrow V$ is continuous. Since $Z \hat{\otimes} \mathbb{C} = Z \otimes \mathbb{C}$, we see that the injectivity of (6.7) for $X = \mathbb{C}$ follows from that of (6.6). Thus, $v \circ i$ is non-degenerate.

(ii) \Rightarrow (iii). This holds by Proposition 6.11 and Example 6.4.

(iii) \Rightarrow (iv). By hypothesis, the map $\text{Vect}(X, V \otimes Y) \rightarrow \text{Vect}(Z \otimes X, W \otimes Y)$,

$$f \mapsto Z \otimes X \xrightarrow{1 \otimes f} Z \otimes V \otimes Y \xrightarrow{i \otimes 1} (Z \hat{\otimes} V) \otimes Y \xrightarrow{v \otimes 1} W \otimes Y$$

is injective for all complex vector spaces X, Y , and we should prove that also the map $\text{Hilb}(X, V \hat{\otimes} Y) \rightarrow \text{Hilb}(Z \hat{\otimes} X, W \hat{\otimes} Y)$,

$$f \mapsto Z \hat{\otimes} X \xrightarrow{1 \hat{\otimes} f} Z \hat{\otimes} V \hat{\otimes} Y \xrightarrow{v \hat{\otimes} 1} W \hat{\otimes} Y \quad (6.8)$$

is injective for all Hilbert spaces X, Y . But the map $\text{Hilb}(Z \hat{\otimes} X, W \hat{\otimes} Y) \rightarrow \text{Vect}(Z \otimes X, W \hat{\otimes} Y)$ is injective, so it will suffice to show that the composite of (6.8) with this last map is injective; and for that, it clearly suffices to prove the case where $X = \mathbb{C}$.

Thus we need to prove that the induced map $V \hat{\otimes} Y \rightarrow \text{Vect}(Z, W \hat{\otimes} Y)$ is injective, or equivalently that for $z \in Z$ the maps

$$V \hat{\otimes} Y \xrightarrow{z \hat{\otimes} -} Z \hat{\otimes} V \hat{\otimes} Y \xrightarrow{v \hat{\otimes} 1} W \hat{\otimes} Y$$

are jointly injective. These are continuous linear maps, so it will suffice to show that the maps

$$\begin{array}{ccccccc}
 V \otimes Y & \xrightarrow{z \otimes -} & Z \otimes V \otimes Y & \xrightarrow{i \otimes 1} & Z \hat{\otimes} V \otimes Y & \xrightarrow{v \otimes 1} & W \otimes Y \\
 \downarrow i & & \downarrow 1 \otimes i & & \downarrow i & & \downarrow i \\
 V \hat{\otimes} Y & \xrightarrow{z \otimes -} & Z \otimes (V \hat{\otimes} Y) & \xrightarrow{i} & Z \hat{\otimes} V \hat{\otimes} Y & \xrightarrow{v \hat{\otimes} 1} & W \hat{\otimes} Y
 \end{array}$$

are jointly injective, or equivalently that the upper horizontals are jointly injective. But this follows from (iii).

(iv) \Rightarrow (i). Put $Y = \mathbb{C}$. □

APPENDIX A. SOME IDENTITIES AND THEIR STRING DIAGRAMMATIC PROOFS

Throughout this Appendix, A will be an object in a braided monoidal category \mathcal{C} , and $t_1, t_2, t_3, t_4: A^2 \rightarrow A^2$, $e_1, e_2: A^2 \rightarrow A^2$ and $j: A \rightarrow I$ are morphisms making A a regular weak multiplier bimonoid in \mathcal{C} . The multiplication

$$\text{multiplication} := \begin{array}{c} \text{cup} \\ \text{cap} \end{array} = \begin{array}{c} \text{cap} \\ \text{cup} \end{array} = \begin{array}{c} \text{cap} \\ \text{cap} \end{array} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{cap} \end{array}$$

is assumed to be non-degenerate with respect to some class of objects containing I , A and A^2 . The following notation is used.

$$\begin{array}{cccc}
 \sqcap_1^R := \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \sqcap_2^R := \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \sqcap_1^L := \begin{array}{c} \text{cup} \\ \text{cup} \end{array} & \sqcap_2^L := \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \\
 \sqcap_1^L := \begin{array}{c} \text{cup} \\ \text{cup} \end{array} & \sqcap_2^L := \begin{array}{c} \text{cup} \\ \text{cup} \end{array} & \sqcap_1^R := \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \sqcap_2^R := \begin{array}{c} \text{cap} \\ \text{cap} \end{array}
 \end{array}$$

A.1. For any morphisms $XA \xrightarrow{f_1} A \xleftarrow{f_2} AX$ satisfying (2.1), the following identities hold.

$$\begin{array}{ll}
 \text{(i)} & \begin{array}{cccc} \begin{array}{c} \text{cap} \\ \text{cap} \end{array} = \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \begin{array}{c} \text{cap} \\ \text{cap} \end{array} = \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \begin{array}{c} \text{cup} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} & \begin{array}{c} \text{cup} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \end{array} \\
 \text{(ii)} & \begin{array}{cccc} \begin{array}{c} \text{cap} \\ \text{cap} \end{array} = \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \begin{array}{c} \text{cap} \\ \text{cap} \end{array} = \begin{array}{c} \text{cap} \\ \text{cap} \end{array} & \begin{array}{c} \text{cup} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} & \begin{array}{c} \text{cup} \\ \text{cup} \end{array} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \end{array}
 \end{array}$$

These identities hold in particular if both f_1 and f_2 are equal to the multiplication of A .

Proof. Note that the identities of part (i) hold for any morphisms satisfying the conditions of Lemma 2.3; and the identities of part (ii) hold for any morphisms obeying (2.18). We prove one identity in each group; all the others follow symmetrically.

(i) The first identity follows by non-degeneracy of the multiplication and

(ii) Similarly, the first identity follows by non-degeneracy of the multiplication and

□

A.2.

Proof. The stated equality is obtained composing by a suitable braiding isomorphism the equal expressions in

□

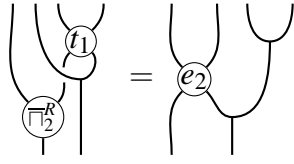
A.3.

Proof. Note that the first equality of the claim holds, in fact, for any weakly counital fusion morphism (t_1, e_1, j) , by the following reasoning.

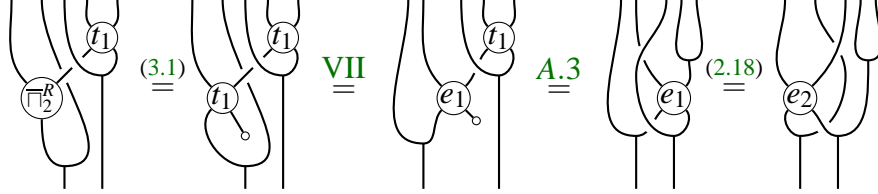
By the first equality of the claim and by its coopposite

proving the second equality of the claim.

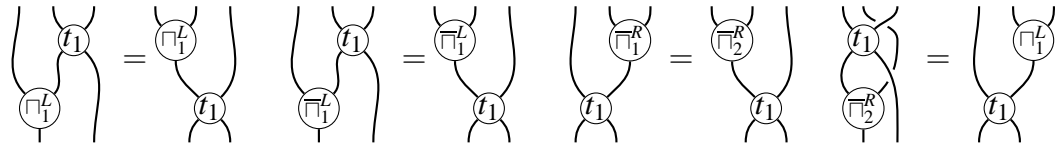
□

A.4. 

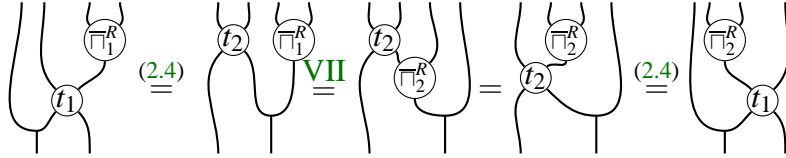
Proof. Note that this holds, in fact, for any weakly counital fusion morphism (t_1, e_1, j) and a morphism e_2 obeying (2.18). It follows by the non-degeneracy of the multiplication and



□

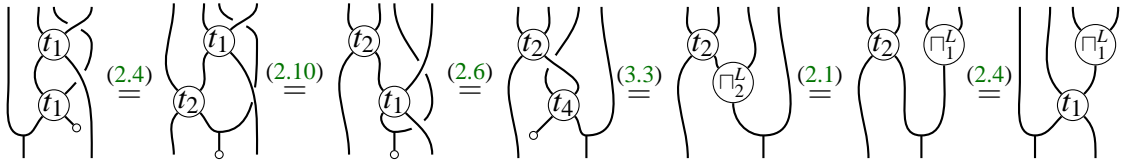
A.5. 

Proof. The first identity of the claim follows from Axiom V on post-composing with $j11$, and the second follows from (2.10) on post-composing with $j11$. The third identity follows by the non-degeneracy of the multiplication and the calculation

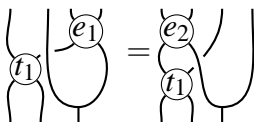


in which the unlabelled equality follows from the opposite-coopposite of the second identity of the claim.

The last identity of the claim follows by the calculation



and the non-degeneracy of the multiplication. □

A.6. 

Proof. Note that the claim holds for any weakly counital fusion morphism (t_1, e_1, j) and a morphism e_2 obeying (2.18). It follows by the calculation

and the non-degeneracy of the multiplication. \square

A.7.

Proof. This follows by Axiom III and (2.14). \square

A.8.

Proof. The first identity of the claim follows by the non-degeneracy of the multiplication and

where the unlabelled equality holds by the opposite of Axiom V. The second identity of the claim follows by

\square

A.9. For the morphisms $d_1 := \text{diagram with } t_1$ and $d_2 := \text{diagram with } t_2$, the following hold.

The second equality says that d_1 and d_2 are the components of an \mathbb{M} -morphism $d: A \rightarrow A^2$, which can be regarded as a generalized (multiplier-valued) comultiplication; the third says that this \mathbb{M} -morphism is multiplicative in the sense of [6].

Proof. The first identity follows by

Using this together with the associativity of the multiplication we obtain

that is, the second identity of the claim. The third identity of the claim follows by

The penultimate identity of the claim is immediate by Axiom IV and the last one follows by the first identity in part (ii) of A.1 and Axiom III. \square

A.10.

Proof. The first identity of the claim follows by the non-degeneracy of the multiplication and

in which the unlabelled equality follows from the coopposite of the third displayed identity of A.5 on post-composing with 1_j . The second identity is equivalent, via non-degeneracy of the multiplication and repeated use of (2.1), to

which in turn follows from the last equality of A.5.

The last identity is equivalent, via repeated use of (2.1), to

which in turn is a straightforward consequence of (2.11). \square

A.11.

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}.$$

Proof. This follows by post-composing with j_1 from the calculation

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \stackrel{\text{A.1}}{=} \stackrel{\text{A.3}}{=} \stackrel{\text{A.5}}{=} \stackrel{\text{A.10}}{=} \stackrel{\text{A.3}}{=} \stackrel{\text{A.1}}{=}$$

and non-degeneracy of the multiplication. \square

A.12.

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array}$$

Proof. The first identity of the claim follows by

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \stackrel{\text{A.1}}{=} \stackrel{(2.14)}{=} \stackrel{\text{A.1}}{=}$$

and the second is similar, using (2.17) in place of (2.14). \square

A.13.

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}$$

Proof. The first identity follows by the non-degeneracy of the multiplication and the equality of the second and last expressions in the calculation

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \stackrel{(2.5)}{=} \stackrel{(2.2)}{=} \stackrel{(2.4)}{=} \stackrel{(2.9)}{=} \stackrel{(3.3)}{=} \stackrel{(2.2)}{=}$$

while the second identity follows from the associativity and non-degeneracy of the multiplication, and the equality of the first and penultimate expressions. \square

A.14.

Proof. The first assertion holds by

and the second follows similarly on applying (2.4) twice. □

A.15.

Proof. This follows from the calculation

and non-degeneracy and associativity of the multiplication. □

A.16.

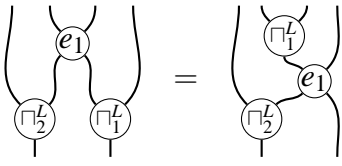
Recall that in the definitions of the papers [24, 25, 4] this is imposed as an axiom.

Proof. This follows by the non-degeneracy of the multiplication and

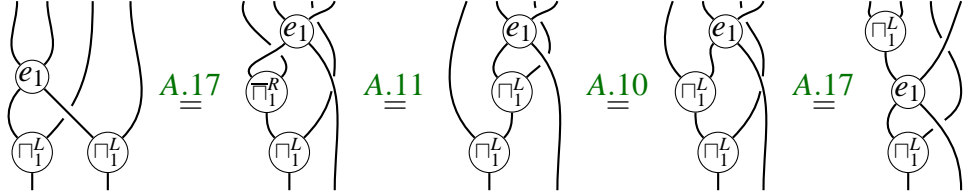
where the equalities labelled A.3 also use the coopposite of A.3. □

A.17.

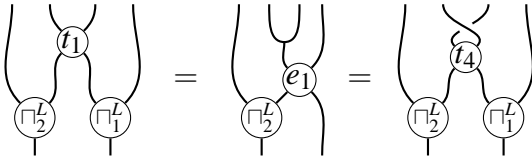
Proof. Immediate from A.16. □

A.18. 

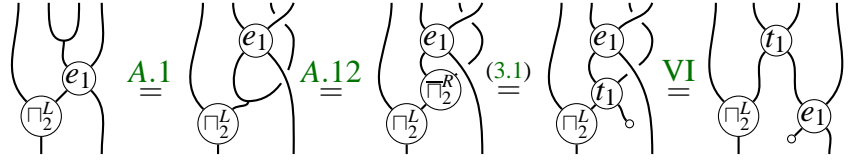
Proof. By the non-degeneracy of the multiplication with respect to A and (2.1), the claim is equivalent to



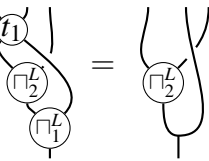
□

A.19. 

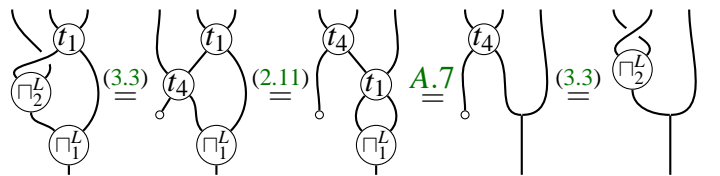
Proof. The second equality of the claim is immediate by the coopposite of the first identity of A.3. The first equality of the claim follows by



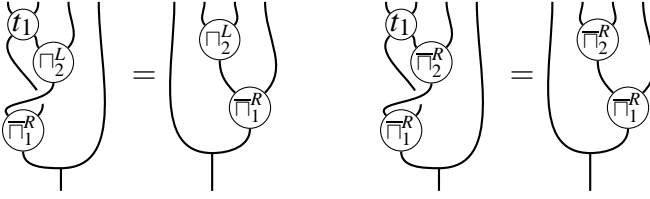
□

A.20. 

Proof. This follows by pre-composing with a suitable braid isomorphism the equal outermost expressions of



□

A.21. 

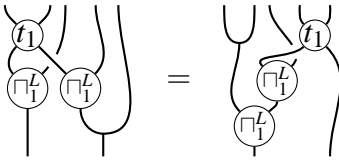
Proof. From the coopposites of the identities of A.12 and multiple uses of (2.1) we obtain

$$\begin{array}{c} \pi_2^L \\ \downarrow \\ \pi_1^R \end{array} = \begin{array}{c} \cup \\ \downarrow \\ \pi_1^R \end{array} = \begin{array}{c} \pi_2^R \\ \downarrow \\ \pi_1^R \end{array}.$$

With their help both equalities of the claim become equivalent to the equality of the outermost expressions in

$$\begin{array}{c} t_1 \\ \downarrow \\ \pi_1^R \end{array} \stackrel{\text{A.1}}{=} \begin{array}{c} t_1 \\ \downarrow \\ e_1 \end{array} \stackrel{\text{III}}{=} \begin{array}{c} \pi_2^R \\ \downarrow \\ \pi_1^R \end{array} \stackrel{(2.1)}{=} \begin{array}{c} \pi_1^R \end{array}.$$

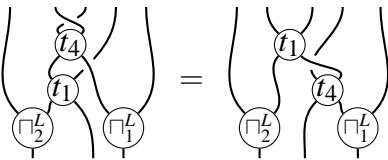
□

A.22. 

Proof. This follows by

$$\begin{array}{c} t_1 \\ \downarrow \\ \pi_1^L \end{array} \stackrel{\text{A.11}}{=} \begin{array}{c} t_1 \\ \downarrow \\ \pi_1^R \end{array} \stackrel{\text{A.3}}{=} \begin{array}{c} e_1 \\ \downarrow \\ \pi_1^L \end{array} \stackrel{\text{A.1}}{=} \begin{array}{c} e_1 \\ \downarrow \\ \pi_1^L \end{array} = \begin{array}{c} t_1 \\ \downarrow \\ \pi_1^L \end{array}$$

where the unlabelled equality is obtained using an equivalent form of the first identity in A.19. □

A.23. 

Proof. This follows by

$$\begin{array}{c} t_4 \\ \downarrow \\ t_1 \\ \downarrow \\ \pi_2^L \end{array} \stackrel{\text{A.3}}{=} \begin{array}{c} e_1 \\ \downarrow \\ t_1 \\ \downarrow \\ \pi_2^L \end{array} \stackrel{\text{VI}}{=} \begin{array}{c} t_1 \\ \downarrow \\ e_1 \\ \downarrow \\ \pi_2^L \end{array} \stackrel{\text{A.15}}{=} \begin{array}{c} t_1 \\ \downarrow \\ e_1 \\ \downarrow \\ \pi_2^L \end{array} \stackrel{\text{A.1}}{=} \begin{array}{c} t_1 \\ \downarrow \\ e_1 \\ \downarrow \\ \pi_2^L \end{array} \stackrel{\text{A.3}}{=} \begin{array}{c} t_1 \\ \downarrow \\ t_4 \\ \downarrow \\ \pi_2^L \end{array}$$

where the equalities labelled by A.3 use this in its coopposite form. □

A.24.

Proof. This follows by

□

REFERENCES

- [1] G. Böhm, *Comodules over weak multiplier bialgebras*, Int. J. Math. 25 (2014), 1450037. [12](#), [31](#)
- [2] G. Böhm, S. Caenepeel and K. Janssen, *Weak bialgebras and monoidal categories*, Comm. Algebra 39 (2011), no. 12 (special volume dedicated to Mia Cohen), 4584–4607. [12](#)
- [3] G. Böhm and J. Gómez-Torrecillas, *Firm Frobenius monads and firm Frobenius algebras*, Bull. Math. Soc. Sci. Math. Roumanie 56(104) (2013), no. 3, 281–298. [18](#)
- [4] G. Böhm, J. Gómez-Torrecillas and E. López-Centella, *Weak multiplier bialgebras*, Trans. Amer. Math. Soc. 367 (2015), 8681–8721. [1](#), [2](#), [3](#), [12](#), [14](#), [18](#), [19](#), [31](#), [34](#), [36](#), [52](#)
- [5] G. Böhm and S. Lack, *Multiplier bialgebras in braided monoidal categories*, Journal of Algebra 423 (2015), 853–889. [1](#), [2](#), [6](#), [12](#), [14](#), [34](#), [35](#)
- [6] G. Böhm and S. Lack, *A category of multiplier bimonoids*, Applied Categorical Structures, to appear; available as <http://arxiv.org/abs/1509.07171>. [2](#), [4](#), [39](#), [49](#)
- [7] G. Böhm and S. Lack, *A simplicial approach to multiplier bimonoids*, Preprint available at <http://arxiv.org/abs/1512.01259>. [2](#)
- [8] G. Böhm and S. Lack, *Multiplier Hopf monoids*, Preprint available at <http://arxiv.org/abs/1511.03806>. [2](#)
- [9] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras. I. Integral theory and C*-structure*, J. Algebra 221 (1999), no. 2, 385–438. [2](#)
- [10] G. Böhm and K. Szlachányi, *Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace*, J. Algebra 233 (2000), 156–212. [12](#)
- [11] G. Böhm and J. Vercruysse, *Morita theory for comodules over corings*, Comm. Algebra 37 (2009), no. 9, 3207–3247. [18](#)
- [12] T. Brzeziński, L. Kadison and R. Wisbauer, *On coseparable and biseparable corings*, in: “Hopf Algebras in Noncommutative Geometry and Physics”, S. Caenepeel and F. Van Oystaeyen (eds.), Monographs on pure and applied mathematics vol. 239 pp 71–88, Marcel Dekker New York, 2004. [12](#), [18](#)
- [13] J. Dauns, *Multiplier rings and primitive ideals*, Trans. Amer. Math. Soc. 145 (1969), 125–158. [40](#)
- [14] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Annals of Mathematics. Second Series 162 (2005): 581–642. [2](#)
- [15] H. Hogbe-Nlend, *Bornologies and functional analysis*. Translated from the French by V. B. Moscatelli. North-Holland Mathematics Studies, Vol. 26. Notas de Matemática, No. 62. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. [1](#), [42](#)
- [16] P.T. Johnstone, *Adjoint Lifting Theorems for Categories of Algebras*, Bull. London Math. Soc. 7 (1975) no. 3, 294–297. [41](#)
- [17] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras, vol. I*. Academic Press, New York, 1983. [13](#)
- [18] R. Meyer, *Local and Analytic Cyclic Homology*. EMS Tracts in Mathematics Vol. 3, 2007. [1](#), [13](#), [42](#)
- [19] F. Nill, *Axioms for weak bialgebras*, preprint available at <http://arxiv.org/abs/math/9805104>. [2](#)

- [20] C. Pastro and R. Street, *Weak Hopf monoids in braided monoidal categories*, Algebra and Number Theory 3 (2009) no. 2, 149-207. [12](#)
- [21] D. Quillen, *Module theory over nonunital rings*, Unpublished Notes, 1997. [18](#), [19](#)
- [22] A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc. 342 (1994), 917-932. [2](#)
- [23] A. Van Daele, *Separability Idempotents and Multiplier Algebras*, preprint available at <http://arxiv.org/abs/1301.4398>.
- [24] A. Van Daele and S. Wang, *Weak Multiplier Hopf Algebras. The main theory*, Journal für die reine und angewandte Mathematik (Crelles Journal) 705 (2015), 155-209. [2](#), [52](#)
- [25] A. Van Daele and S. Wang, *Weak Multiplier Hopf Algebras. Preliminaries, motivation and basic examples*, in: Operator Algebras and Quantum Groups, W. Pusz and P.M. Sołtan (eds.), Banach Center Publ. 98 (2012), 367-415. [2](#), [36](#), [52](#)
- [26] C. Voigt, *Bornological quantum groups*, Pacific J. Math. 235 (2008) no. 1, 93-135. [42](#)
- [27] B. Zimmermann-Huisgen, *Pure submodules of direct products of free modules*, Math. Annalen 224 (1976) no. 2, 233-245. [41](#)

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